

Hamiltonian paths, unit-interval complexes, and determinantal facet ideals

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April 12, 2021

Abstract

We study d -dimensional generalizations of three mutually related topics in graph theory: Hamiltonian paths, (unit) interval graphs, and binomial edge ideals. We provide partial high-dimensional generalizations of Ore and Pósa’s sufficient conditions for a graph to be Hamiltonian. We introduce a hierarchy of combinatorial properties for simplicial complexes that generalize unit-interval, interval, and co-comparability graphs. We connect these properties to the already existing notions of determinantal facet ideals and (tight and weak) Hamiltonian paths in simplicial complexes. Some important consequences of our work are:

- (1) Every unit-interval strongly-connected d -dimensional simplicial complex is traceable.
(This extends the well-known result “unit-interval connected graphs are traceable”.)
- (2) Every unit-interval d -complex that remains strongly connected after the deletion of d or less vertices, is Hamiltonian.
(This extends the fact that “unit-interval 2-connected graphs are Hamiltonian”.)
- (3) Unit-interval complexes are characterized, among traceable complexes, by the property that the minors defining their determinantal facet ideal form a Gröbner basis for a diagonal term order which is compatible with the traceability of the complex.
(This corrects a recent theorem by Ene et al., extends a result by Herzog and others, and partially answers a question by Almousa–Vandebogert.)
- (4) Only the d -skeleton of the simplex has a determinantal facet ideal with linear resolution.
(This extends the result by Kiani and Saeedi-Madani that “only the complete graph has a binomial edge ideal with linear resolution”.)
- (5) The determinantal facet ideals of all under-closed and semi-closed complexes have a square-free initial ideal with respect to lex. In characteristic p , they are even F-pure.

Introduction

The first Combinatorics paper in History is apparently Leonhard Euler’s 1735 solution of the Königsberg bridge problem. In that article, Euler introduced the notion of graph, and studied cycles (now called ‘Eulerian’) that touch all edges exactly once. Euler proved that the graphs admitting them, are exactly those graphs with all vertices of even degree. *Hamiltonian* cycles

*Supported by NSF Grant 1855165, “Geometric Combinatorics and Discrete Morse Theory”.

**Supported by 100021 2019 ADS FRA, “Algebra commutativa: ideali iniziali square-free, singolarità in caratteristica mista e aspetti computazionali”.

are instead cycles that touch all *vertices* exactly once; they are named after sir William Rowan Hamilton, who in 1857 invented a puzzle game which asked to find one such cycle in the icosahedron. Unlike for the Eulerian case, figuring out if a graph admits a Hamiltonian cycle or not is a hard problem, now known to be NP-complete [Kar72].

Even if simple characterizations are off the table, in the 1950s and 1960s Dirac, Ore, Pósa and others were able to obtain simple conditions on the vertex degrees (in the spirit of Euler’s work) that are *sufficient* for a graph to admit Hamiltonian cycles [Dir52, Ore60, Pósa62]. Ore’s theorem, for example, says, “*Any graph with n vertices such that $\deg u + \deg v \geq n$ for all non-adjacent vertices u, v , admits a Hamiltonian cycle*”. Ore’s condition is far from being necessary: In any cycle, no matter how large, one has $\deg u + \deg v = 4$ for all u, v .

In the same years, the two papers [LB62] and [GH64] initiated the study of *unit-interval graphs*. This very famous class consists, as the name suggests, of all intersection graphs of a bunch of length-one open intervals on the real line. (That is, we place a node in the middle of each interval, and we connect two nodes with an arc if and only if the corresponding intervals overlap). Bertossi’s theorem says that if they are connected, such graphs always admit *Hamiltonian paths*, i.e. paths that touch all vertices once [Ber83]. Chen–Chang–Chang’s theorem states that 2-connected unit-interval graphs admit Hamiltonian cycles [CCC97]. For these results, the length-one request can be weakened to “pairwise not-nested”, but it cannot be dismissed: Within the larger world of *interval graphs*, one encounters connected graphs such as $K_{1,3}$ that do not admit Hamiltonian paths, and also 2-connected graphs like the G_5 of Remark 45 that do not admit Hamiltonian cycles.

In the 1970s, the work of Stanley and Reisner established a fundamental bridge between Combinatorics and Commutative Algebra, namely, a natural bijection between labeled simplicial complexes on n vertices and radical monomial ideals in a polynomial ring with n variables. This correspondence lead Stanley to prove the famous Upper Bound Theorem for triangulated spheres [Sta14]. After this success, many authors have investigated ways to encode graphs into monomial ideals. In 2010, Herzog et al. [H&10] first considered a natural way to encode graphs into binomial ideals, the so-called *binomial edge ideals*. The catch is that all such binomial edge ideals are radical [H&10]. In the process, Herzog et al. re-discovered unit-interval graphs, characterizing them as the graphs whose binomial edge ideals have quadratic Gröbner bases with respect to a diagonal term order [H&10, Theorem 1.1].

So far, we sketched three graph-theoretic topics from three different centuries: Hamiltonian paths, (unit) interval graphs, binomial edge ideals. In the last years, motivated by the many applications produced by the above one dimensional concepts, there has been an increasing interest in expanding these three notions to higher dimensions. Specifically:

- Katona–Kierstead [KK99] and many others [HS10, K&10, RSR08] have studied “tight Hamiltonian paths” and “loose Hamiltonian paths” in d -dimensional simplicial complexes; both notions for $d = 1$ boil down to ordinary Hamiltonian paths. The good news is that extremal combinatorics provides a non-trivial way to extend Dirac’s theorem for d -complexes with a very large number of vertices that satisfy certain ridge-degree conditions. The bad news is that already Ore and Pósa’s theorems seem very hard to extend.
- Ene et al. [E&13] introduced “determinantal facet ideals”, which directly generalize binomial edge ideals, and “closed d -complexes”, which generalize ‘unit-interval graphs’. The good news is that the definitions are rather natural. The bad news is that determinantal facet ideals are not radical in general (see Example 73), and they are hard to manipulate; alas, the two main results of the paper [E&13] are incorrect, cf. Remark 85.

In the present paper we take a new, unified look at these approaches. In Chapter 1, we introduce a notion of ‘weakly-Hamiltonian paths’ for d -dimensional simplicial complexes that

for $d = 1$ also boils down to ordinary Hamiltonian paths. This weaker notion enables us to obtain a first, partial extension of Dirac, Ore and Pósa’s theorem to higher dimensions:

Main Theorem I (Higher-dimensional Ore and Dirac, cf. Proposition 18 and Corollary 20). *Let Δ be any traceable d -complex on $n > 2d$ vertices. If in some labeling that makes Δ traceable the two $(d-1)$ -faces σ and τ formed by the first d and the last d vertices, respectively, have facet degrees summing up to at least n , then Δ admits a weakly-Hamiltonian cycle. In particular, if in a traceable pure d -complex with n vertices, every $(d-1)$ -face belongs to at least $\frac{n}{2}$ facets, then the complex admits a weakly-Hamiltonian cycle.*

Main Theorem II (Higher-dimensional Pósa, cf. Proposition 23). *Let Δ be any traceable pure d -complex on n vertices, $n > 2d$. Suppose that with any labeling in which Δ has a weakly-Hamiltonian path, Δ is traceable. Let $\sigma_1, \sigma_2, \dots, \sigma_s$ be the $(d-1)$ -faces of Δ^* , ordered so that $d_1 \leq d_2 \leq \dots \leq d_s$, where $d_i \stackrel{\text{def}}{=} d_{\sigma_i}$ is the number of d -faces containing σ_i . If for every $d \leq k < \frac{n}{2}$ one has $d_{k-d+1} > k$, then Δ admits a weakly-Hamiltonian cycle.*

As you can see these results are conditional: ‘Traceability’, i.e. the existence of a *tight* Hamiltonian path, must be known a priori, in order to infer the existence of a *weakly*-Hamiltonian cycle. This sounds like a bad deal, but in the one-dimensional case our results above still immediately imply the original theorems by Ore and Pósa for graphs. Moreover, since no extremal combinatorics is used in the proof, there is an advantage: Main Theorems I and II do not require the number of vertices to be extremely large. On the contrary: In the two-dimensional case, they already apply to complexes with five vertices.

In Chapter 2, we introduce a hierarchy of four natural properties that progressively weaken (for strongly-connected complexes) the notion of “closed d -complexes”, as originally proposed in [E&13]. We introduce “unit-interval”, “under-closed”, and “weakly-closed” complexes, as natural combinatorial higher-dimensional generalizations of unit-interval graphs, of interval graphs, and of co-comparability graphs, respectively. The forth property, called “semi-closed”, is intermediate between “under-closed” and “weakly-closed”; it is also defined very naturally, but it seems to be new already for graphs. We will see its algebraic consequence in Main Theorem VI below. The main goal of Chapter 2 is to connect this hierarchy to the notions of Chapter 1:

Main Theorem III (Higher-dimensional Bertossi, Theorem 56). *Every unit-interval strongly-connected d -dimensional simplicial complex is traceable.*

Main Theorem IV (Higher-dimensional Chen–Chan–Chang, Theorem 60). *Every unit-interval d -dimensional simplicial complex that remains strongly connected after the deletion of d or less vertices, however chosen, is Hamiltonian.*

Finally, Chapter 3 is dedicated to the connection with commutative algebra. For a homogeneous ideal of polynomials, having a square-free Gröbner degeneration is a strong and desirable property. In 2020, Conca and the third author proved Herzog’s conjecture that if a homogeneous ideal I has a square-free initial ideal $\text{in}(I)$, then the extremal Betti numbers of I and $\text{in}(I)$ are the same [CV20]. This allows us to infer the depth, the Castelnuovo–Mumford regularity, and many other invariants of the ideals I with squarefree initial-ideal, simply by computing these invariants on the initial ideal — which is a much simpler task, because the aforementioned Stanley–Reisner correspondence activates techniques from combinatorial topology. Building on the very recent work of the second author [Sec21], we are able to revise one of the results claimed in Ene et al [E&13] as follows:

Main Theorem V (Theorem 82 and 87). *A strongly-connected d -dimensional simplicial complex Δ is unit-interval if and only if the complex is traceable and with respect to the same labeling, the minors defining the determinantal facet ideal of Δ form a Gröbner basis with respect to any diagonal term order.*

We conclude our work with a result that provides a broad class of determinantal facet ideals that are radical, and even F -pure (if the characteristic is positive):

Main Theorem VI (Theorem 77). *The determinantal facet ideals of all semi-closed complexes are radical. Indeed, they have a square-free initial ideal with respect to any diagonal term order. Moreover, in characteristic $p > 0$, the quotients by these ideals are all F -pure.*

The proof relies once again on the recent work by the second author [Sec21]. Since all shifted complexes are under-closed, and in particular semi-closed, Theorem 77 immediately implies that the determinantal facet ideals of shifted complexes admit a square-free Gröbner degeneration and, in positive characteristic, define F -pure rings. As a consequence of Main Theorem VI, we can extend to all dimensions the result by Kiani and Saeedi-Madani that “among all graphs, only complete graphs have a binomial edge ideal with a linear resolution” [SK12]. Namely, we prove that among all d -dimensional simplicial complexes with n vertices, only the d -skeleta of simplices have a determinantal facet ideal with a linear resolution (Corollary 81).

Notation

Throughout d, n are positive integers, with $d < n$. We denote by Σ^d the d -simplex, and by Σ_n^d the d -skeleton of Σ^{n-1} . We write each face of Σ_n^d by listing its vertices in increasing order. We describe simplicial complexes by listing their facets in any order, e.g. $\Delta = 123, 124, 235$. For any d -face $F = a_0 a_1 \cdots a_d$ of Σ_n^d , we call *gap of F* the integer $\text{gap}(F) \stackrel{\text{def}}{=} a_d - a_0 - d$, which counts the integers i strictly between a_0 and a_d that are not present in F . For each i in $\{1, 2, \dots, n-d\}$, we call H_i the d -face of Σ_n^d with vertices $i, i+1, \dots, i+d$. Clearly, H_1, H_2, \dots, H_{n-d} are exactly those faces of Σ_n^d that have gap zero. With abuse of notation, we extend the definition of H_i also to $i \in \{n-d+1, \dots, n\}$ using “congruence modulo n ”. Namely, by “ $n+1$ ” we mean vertex 1, by “ $n+2$ ” we mean vertex 2, and so on. So H_n will be the d -face adjacent to H_1 and of vertices $\{n, 1, 2, 3, \dots, d\}$, which we write down in increasing order, so $H_n = 123 \cdots d n$. Note that $\text{gap}(H_i) > 0$ when $i > n-d$.

Definition 1 (traceable, Hamiltonian). A complex Δ is (*tight-*) *traceable* if it has a labeling such that H_1, \dots, H_{n-d} are in Δ . It is (*tight-*) *Hamiltonian* if it has a labeling such that all of H_1, \dots, H_n are in Δ .

Clearly, Hamiltonian implies traceable. For $d = 1$, Definition 1 boils down to the classical notions of traceable and Hamiltonian graphs, that is, graphs that admits a Hamiltonian path and a Hamiltonian cycle, respectively. In fact, nobody prevents us from relabeling the vertices in the order in which we encounter them along such path (or cycle).

Recall that two facets of a pure simplicial d -complex are *adjacent* if their intersection has cardinality d , or equivalently, dimension $d-1$. For example, each H_i is adjacent to H_{i+1} . The *dual graph* of a pure simplicial d -complex Δ has nodes corresponding to the facets of Δ ; two nodes are connected by an arc if and only if the corresponding facets of Δ are adjacent. A pure simplicial d -complex Δ is *strongly-connected* if its dual graph is connected. For $d \geq 1$, every strongly-connected d -complex is connected, and when $d = 1$ the two notions coincide. According to our convention, all strongly-connected simplicial complexes are pure.

Remark 2. The statement “the dual graph of any Hamiltonian d -complex is Hamiltonian” holds true only for $d = 1$: For example, the Hamiltonian simplicial complex

$$\Delta_1 = 123, 234, 345, 456, 567, 678, 789, 189, 129, 147$$

is not even strongly connected, because the facet 147 is isolated in the dual graph. The deletion of vertex 1 from Δ_1 yields a simplicial complex that is not even pure.

1 Weakly-traceable/Hamiltonian complexes and ridge degrees

In this section, we introduce two weaker notions of traceability and Hamiltonicity that first appeared in [K&10], and we study their nontrivial relationship with the “ridge degree”, i.e. how many d -faces contain any given $(d - 1)$ -face. This relationship has a long history, beginning in 1952 with one of the most classical results in graph theory, due to Gabriel Dirac [Dir52], the son of Nobel Prize physicist Paul Dirac:

Theorem 3 (Dirac [Dir52]). *Let G be a graph with n vertices. If $\deg v \geq \frac{n}{2}$ for every vertex v , then G is Hamiltonian.*

Later Øystein Ore [Ore60] improved Dirac’s result and extended it to traceable graphs:

Theorem 4 (Ore [Ore60]). *Let G be a graph with n vertices.*

- (A) *If $\deg u + \deg v \geq n$ for all non-adjacent vertices u, v , the graph G is Hamiltonian.*
- (B) *If $\deg u + \deg v \geq n - 1$ for all non-adjacent vertices u, v , the graph G is traceable.*

Two years later Pósa extended Ore’s condition (A) much further:

Theorem 5 (Pósa [Pós62]). *Let G be a graph with n vertices. Order the vertices v_1, \dots, v_n so that the respective degrees are weakly increasing, $d_1 \leq d_2 \leq \dots \leq d_n$.*

- (C) *If for every $k < \frac{n}{2}$ one has $d_k > k$, the graph G is Hamiltonian.*

These theorems have been generalized in five main directions, over the course of more than a hundred papers (see also Li [Li13] for a survey with a different perspective than ours):

1. Bondy and Chvátal [Bon69, Bo71a, Chv84, BC71] weakened the antecedent in the implication (C) of Pósa’s theorem (see [Far99] for an application to self-complementary graphs);
2. Bondy [Bo71b] strengthened the conclusion of Ore’s theorem, from Hamiltonian to *pancyclic* (=containing cycles of length ℓ for any $3 \leq \ell \leq n$); later Schmeichel–Hakimi [SH74] showed that Pósa and Chvátal’s theorems can be strengthened in the same direction;
3. Fan [Fan84] showed that for 2-connected graphs, it suffices to check Ore’s condition for vertices u and v at distance 2; and even more generally, it suffices to check that for any two vertices at distance two, at least one of them has degree $\geq \frac{n}{2}$. With these weaker assumptions he was still able to achieve a pancyclicity conclusion. See [BCS93], [LLF07], [CSZ14] for recent extensions of Fan’s work.
4. A fourth line of generalizations of Ore’s theorem involved requiring certain vertex sets to have large neighborhood unions, rather than large degrees: Compare Broersma–van den Heuvel–Veldman [BHV93] and Chen–Schelp [CS92].

Here we are interested in the *fifth* main direction, namely, the generalization to higher dimensions. This is historically a rather difficult task: As of today, no straightforward extension of Ore’s theorem or of Pósa’s theorem is known. In 1999 Katona and Kierstead [KK99] gave a generalization of Dirac’s theorem, proving that all pure d -complexes in which every $(d - 1)$ -face is in at least $\frac{2d+1}{2d+2} \cdot n + 3 - d$ facets, is Hamiltonian. Unlike the bound by Dirac, this bound is “far from tight” [KK99]. Using extremal graph theory, Rödl, Szemerédi, and Ruciński [RSR08] were able in 2008 to prove a tighter but ‘extremal’ bound:

Theorem 6 (Rödl–Szemerédi–Ruciński [RSR08]). *For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer n_0 such that every d -dimensional simplicial complex with more than n_0 vertices and in which every $(d - 1)$ -face is in at least $n(\frac{1}{2} + \varepsilon)$ facets is Hamiltonian.*

Now we are ready to introduce the main definition of the present section. Recall that two facets of a pure simplicial d -complex are *incident* if their intersection is nonempty.

Definition 7 (weakly-traceable, weakly-Hamiltonian). A d -dimensional simplicial complex Δ is *weakly-traceable* if it has a labeling such that Δ contains faces H_{i_1}, \dots, H_{i_k} from $\{H_1, \dots, H_{n-d}\}$ that altogether cover all vertices, and such that H_{i_j} is incident to $H_{i_{j+1}}$ for each $j \in \{1, \dots, k-1\}$. In this case, we call the list H_{i_1}, \dots, H_{i_k} a *weakly-Hamiltonian path*.

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Remark 8. These notions are not new. For what we called “weakly-Hamiltonian”, Keevash et al. [K&10] use the term “generic Hamiltonian”. Their paper [K&10] focuses however on the stronger notion of “loose-Hamiltonian” complexes, which are weakly-Hamiltonian complexes where all of the intersections $H_{i_j} \cap H_{i_{j+1}}$ consist of a single point (with possibly one exception). By definition, all Hamiltonian complexes are loose-Hamiltonian, and all loose-Hamiltonian complexes are weakly-Hamiltonian. For $d = 1$ all these different notions converge: “Weakly-Hamiltonian 1-complexes” are simply “graphs with a Hamiltonian cycle”, and “weakly-traceable 1-complexes” are “graphs with a Hamiltonian path”. In 2010 Han–Schacht [HS10] and independently Keevash et al. [K&10] proved the following extension of Theorem 6 above:

Theorem 9 (Han-Schacht [HS10], Keevash et al. [K&10]). *For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer n_0 such that every d -dimensional simplicial complex Δ with more than n_0 vertices, and such that every $(d-1)$ -face of Δ is in at least $n(\frac{1}{2d} + \varepsilon)$ facets, is loose-Hamiltonian, and in particular weakly-Hamiltonian.*

Remark 10. In Definition 7, note that if Δ is weakly-traceable, necessarily $i_1 = 1$ and $i_k = n-d$, because otherwise 1 and n would not be covered. So equivalently, in Def. 7 we could demand

$$\{i_2, \dots, i_{k-1}\} \subset \{2, \dots, n-d-1\}.$$

Note also that if a labeling v_1, \dots, v_n makes Δ (weakly-) traceable, so does the “reverse labeling” v_n, \dots, v_1 . As for Hamiltonian complexes: If a labeling v_1, \dots, v_n makes Δ weakly-Hamiltonian, so does its reverse, and also v_{i_1}, \dots, v_{i_n} , where (i_1, \dots, i_n) is any cyclic permutation of $(1, \dots, n)$. So we may assume that $i_1 = 1$. Or we may assume that $i_k = n-d$. But as the next remark shows, we cannot assume both.

Remark 11. When $d > 1$, not all weakly-Hamiltonian d -complexes are weakly-traceable. For $d = 2$, a simple counterexample is given by

$$\Delta_0 = 123, 156, 345.$$

The weakly-Hamiltonian cycle is of course H_1, H_3, H_5 . Any labeling that makes Δ_0 weakly-Hamiltonian is either the reverse or a cyclic shift (or both) of the labeling above. For parity reasons, in any labeling that makes Δ_0 weakly-Hamiltonian, only one of H_1 and H_4 is in Δ_0 .

Remark 12. Weakly-traceable complexes are obviously connected. Weakly-Hamiltonian complexes are even 2-connected, in the sense that the deletion of any vertex leaves them connected. The converses are well-known to be false already for $d = 1$. In fact, let $n \geq 4$. Let A_{n-2} be the edge-less graph on $n-2$ vertices. Let x, y be two new vertices. The “suspension”

$$\text{susp}(A_{n-2}) \stackrel{\text{def}}{=} A_{n-2} \cup \{x * v : v \in A_{n-2}\} \cup \{y * v : v \in A_{n-2}\}$$

is a 2-connected graph on n vertices that is not Hamiltonian for $n \geq 5$, and not even traceable for $n \geq 6$. In higher dimensions, the Δ_3^d of Lemma 44 is d -connected, but neither weakly-traceable nor weakly-Hamiltonian.

We start with a few Lemmas that are easy, and possibly already known; we include nonetheless a proof for the sake of completeness. For the following lemma, a *subword* of a word is a subsequence formed by *consecutive* letters of a word: So for us “word” is a subword of “subword”, whereas “sword” is not.

Lemma 13. *Let $d \geq 2$. If a d -complex Δ is weakly-Hamiltonian (resp. weakly traceable), then for any $k \in \{1, \dots, d\}$ the k -skeleton of Δ is weakly-Hamiltonian (resp. weakly-traceable).*

Proof. Given a weakly-Hamiltonian path/cycle, replace any d -face H_1 with its $(k + 1)$ -letter subwords, ordered lexicographically. The result, up to canceling possible redundancies, will be a weakly-Hamiltonian path/cycle for the k -skeleton. \square

For example: if $d = 3$ and $k = 1$, suppose that a 3-complex on 8 vertices admits the Hamiltonian path

$$1234, \quad 2345, \quad 5678.$$

Then the 1-skeleton admits the Hamiltonian path

$$12, 23, 34, \quad \cancel{23}, \cancel{34}, 45, \quad 56, 67, 78.$$

The next Lemma is an analog to the fact that Hamiltonian complexes are traceable.

Lemma 14. *Let Δ be a d -dimensional complex that has a weakly-Hamiltonian cycle H_{i_1}, \dots, H_{i_k} , with $k \geq 3$. For any j in $\{1, \dots, k\}$, let m_j be the number of vertices of H_{i_j} that are neither contained in $H_{i_{j-1}}$ nor in $H_{i_{j+1}}$ (where by convention $i_{k+1} \stackrel{\text{def}}{=} i_1$).*

- *If $m_j > 0$, the deletion of those m_j vertices from Δ yields a weakly-traceable complex.*
- *If $m_j = 0$, and in addition $H_{i_{j-1}}$ and $H_{i_{j+1}}$ are disjoint, then Δ itself is weakly-traceable.*

Proof. Fix j in $\{1, \dots, k\}$. If $m_j > 0$, the m_j vertices that belong to H_{i_j} and to no other facet of the cycle are labeled consecutively. So up to relabeling the vertices cyclically, we can assume that they are the vertices $n - m_j + 1, n - m_j + 2, \dots, n - 1, n$. Thus the facet in the cycle they all belong to is the last one, H_{i_k} . Now let D be the complex obtained from Δ by deleting these m_j vertices. It is easy to see that

$$H_1 = H_{i_1}, H_{i_2}, \dots, H_{i_{k-1}}$$

is a weakly-Hamiltonian path for D .

The case $m_j = 0$ is similar: Up to relabeling the vertices cyclically, $i_{j+1} = 1$ and thus $j = k$. By assumption $H_{i_{k-1}}$ and H_1 are disjoint. But since $m_k = 0$, and vertex n does not belong to H_1 , it must belong to $H_{i_{k-1}}$. Therefore $H_{i_{k-1}} = H_{n-d}$. So

$$H_1 = H_{i_1}, H_{i_2}, \dots, H_{i_{k-1}}$$

is a weakly-Hamiltonian path for Δ itself. \square

The next Lemma can be viewed as a d -dimensional extension of the fact that the cone over the vertex set of a graph G is a Hamiltonian graph if and only if the starting graph G is traceable.

Lemma 15. *Let Δ be any d -complex on n vertices. Let Σ^{d-1} be the $(d - 1)$ -simplex. Let Γ be the d -complex obtained by adding to Δ a d -face $v * \Sigma^{d-1}$ for every vertex v in Δ . Then*

$$\Delta \text{ is weakly-traceable} \iff \Gamma \text{ is weakly-Hamiltonian.}$$

Proof. “ \Rightarrow ”: If H_{i_1}, \dots, H_{i_k} is a list of facets proving that Δ is weakly-traceable, then the list $H_{i_1}, \dots, H_{i_k}, H_n, H_{n+1}$ shows that Γ is weakly-Hamiltonian.

“ \Leftarrow ”: Pick a labeling that makes Γ weakly-Hamiltonian. By how the complex Γ is constructed, the vertices of Σ^{d-1} must be labeled consecutively; so without loss, we may assume that they are $n+1, \dots, n+d$. Take a weakly-Hamiltonian cycle for Γ and delete from the list all the d -faces containing any vertex whose label exceeds n . \square

Remark 16. The following statements are valid **only for $d = 1$** .

- (i) “ Δ is weakly-traceable $\iff \Delta \cup w * (d-1)\text{-skel}(\Delta)$ is weakly-Hamiltonian.”
- (ii) “Deleting a single vertex from a weakly-Hamiltonian d -complex yields a weakly-traceable complex.”
- (iii) “Deleting (the interior of) any of the H_i ’s from a weakly-Hamiltonian d -complex yields a weakly-traceable complex.”

Simple counterexamples in higher dimensions are:

- (i) $\Delta_1 = 126, 234, 456, 489, 678$ is not weakly-traceable, yet $\Delta_2 \stackrel{\text{def}}{=} \Delta_1 \cup (10 * 2\text{-skel}(\Delta_1))$ admits the weakly-Hamiltonian cycle $234, 456, 678, 8910, 1210$. This is a counterexample to “ \Leftarrow ”. In contrast, the direction “ \Rightarrow ” holds in all dimensions.
- (ii) If from the Δ_2 above we delete vertex 10, we get back to Δ_1 , not weakly-traceable.
- (iii) $\Delta_3 = 1234, 2345, 5678, 16710, 18910$ is weakly-Hamiltonian, but the deletion of (the interior) of 5678 yields a complex that is not weakly-traceable.

Our first non-trivial result is an “Ore-type result”: We shall see later that in some sense it extends ‘most’ of the proof of Ore’s theorem 4, part (A), to all dimensions.

Definition 17. Let Δ be a pure d -dimensional simplicial complex, and let σ be any $(d-1)$ -face of Δ . The *degree* d_σ of σ is the number of d -faces of Δ containing σ .

Proposition 18. *Let Δ be a traceable d -dimensional simplicial complex on n vertices, $n > 2d$. If in some labeling that makes Δ traceable the two $(d-1)$ -faces σ and τ formed by the first d and the last d vertices, respectively, satisfy $d_\sigma + d_\tau \geq n$, then Δ is weakly-Hamiltonian.*

Proof. Since $n > 2d$, the two faces σ and τ are disjoint. Let $J \stackrel{\text{def}}{=} \{d+2, d+3, \dots, n-d\}$. For every i in J , which has cardinality $n-2d-1$, consider the two d -faces of Σ_n^d

$$S_i \stackrel{\text{def}}{=} \sigma * i \quad \text{and} \quad T_i \stackrel{\text{def}}{=} (i-1) * \tau.$$

Now there are two cases, both of which will result in a weakly-Hamiltonian cycle:

CASE 1: For some i , both S_i, T_i are in Δ . We are going to introduce a new vertex labeling ℓ_1, \dots, ℓ_n . The “consecutive facets of the new labeling” will be called $L_1 = \ell_1 \cdots \ell_d \ell_{d+1}$, $L_2 = \ell_2 \cdots \ell_{d+2}$, and so on. The following describes a weakly-Hamiltonian cycle:

- Start with the first $i-1$ vertices in the same order: That is, set $\ell_1 \stackrel{\text{def}}{=} 1, \dots, \ell_{i-1} \stackrel{\text{def}}{=} i-1$. Hence $L_1 = H_1, L_2 = H_2, \dots$, up until $L_{i-d-1} = H_{i-d-1}$, which (since Δ is traceable) is the first of the H_i ’s that contains the vertex $i-1$.
- Then set $L_i \stackrel{\text{def}}{=} T_i$. The vertices of τ are to be relabeled by $\ell_i, \ell_{i+1}, \dots, \ell_{i+d}$: Specifically, label by ℓ_i the vertex that is in H_{n-d} but not in H_{n-d-1} , by ℓ_{i+2} the vertex in H_{n-d-1} but not in H_{n-d-2} , and so on. Facet-wise, we are traveling in reverse order across the *last* facets of the original labeling. Stop until you get to relabel vertex i by ℓ_n . (Or equivalently, if you prefer to think about facets, stop once you reach facet H_i .)
- The weakly-Hamiltonian cycle gets then concluded with S_i , which is adjacent to $L_1 = H_1$ via σ . The facets previously called $H_{i-d}, H_{i-d+1}, \dots, H_{i-1}$ are not part of the new weakly-Hamiltonian cycle.

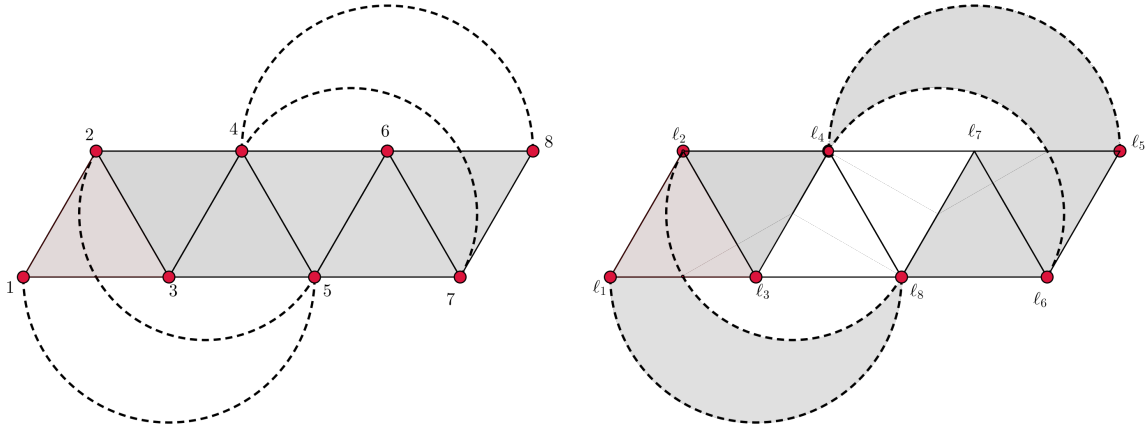


Figure 1: LEFT: The dashed triangles are S_5 and T_5 . Were they both in Δ , then one could relabel the vertices and create a weakly-Hamiltonian cycle (RIGHT).

CASE 2: For all i , at most one of S_i, T_i is in Δ . Since the two sets $\{i \in J : \sigma * i \in \Delta\}$ and $\{i \in J : (i-1) * \tau \in \Delta\}$ are disjoint, the sum of their cardinalities is the cardinality of their union, which is contained in J . So

$$|\{i \in J : \sigma * i \in S\}| + |\{i \in J : (i-1) * \tau \in \Delta\}| \leq |J| = n - 2d + 1. \quad (1)$$

Now, we claim that either $\sigma * n$ or $1 * \tau$ is a face of Δ . From the claim the conclusion follows immediately, as such face creates a weakly-Hamiltonian cycle. We prove the claim by contradiction. Suppose Δ contains neither $\sigma * n$ nor $1 * \tau$. Every d -face containing σ is of the form $\sigma * v$, where v is either in J or in the set $\{d, n-d+1, n-d+2, \dots, n-1\}$ (which has size d). So

$$d_\sigma \leq |\{i \in J : \sigma * i \in \Delta\}| + d. \quad (2)$$

Symmetrically, the d -faces containing τ are of the form $w * \tau$, with w either in J or in the size- d set $\{2, 3, \dots, d, n-d+1\}$. So

$$d_\tau \leq |\{i \in J : (i-1) * \tau \in \Delta\}| + d. \quad (3)$$

Putting together inequalities 1, 2 and 3, we reach a contradiction:

$$d_\sigma + d_\tau \leq (n - 2d - 1) + d + d = n - 1. \quad \square$$

Corollary 19. Let Δ be a traceable d -dimensional simplicial complex on n vertices, $n > 2d$. If for any two disjoint $(d-1)$ -faces σ and τ one has $d_\sigma + d_\tau \geq n$, then Δ is weakly-Hamiltonian.

Corollary 20. Let Δ be a traceable d -dimensional simplicial complex on n vertices, $n > 2d$. If every $(d-1)$ -face of Δ belongs to at least $\frac{n}{2}$ facets of Δ , then Δ is weakly-Hamiltonian.

Example 21. Let $n > 2d$. Let Δ_4 be the simplicial complex on n vertices obtained from Σ_n^d by removing the interior of the d -faces $H_{n-d+1}, H_{n-d+2}, \dots, H_n$. By construction Δ_4 is traceable, but the given labeling (as well as any labeling obtained from it by reversing or cyclic shifting) fails to prove that Δ_4 is weakly-Hamiltonian. Now, in Δ_4 , the $(d-1)$ -faces $\mu_i \stackrel{\text{def}}{=} H_i \cap H_{i+1}$, with $i \in \{n-d+1, n-d+2, \dots, n-1\}$, have degree $n-d-2$. All other $(d-1)$ -faces ν_j contained in one of $H_{n-d+1}, H_{n-d+2}, \dots, H_n$ have degree $n-d-1$. Finally, all $(d-1)$ -faces not contained in any of $H_{n-d+1}, H_{n-d+2}, \dots, H_n$ have degree $n-d$. Therefore:

- If $n \geq 2d + 4$, Corollary 20 tells us that Δ_4 is weakly-Hamiltonian, because $n - d - 2 \geq \frac{n}{2}$.
- If $n = 2d + 3$ or $n = 2d + 2$, any two of the μ_i 's are incident, and any ν_j is incident to all of the μ_i 's. Hence, for any two *disjoint* $(d - 1)$ -faces σ and τ , we do have $d_\sigma + d_\tau \geq 2n - 2d - 2 \geq n$. So we can still conclude that Δ_4 is weakly-Hamiltonian via Corollary 19.
- If $n = 2d + 1$, then the assumptions of Corollaries 20 and 19 are not met, but Proposition 18 is still applicable. In fact, for the facets σ resp. τ formed by the first resp. the last vertices of the given labeling, one has $d_\sigma + d_\tau = (n - d) + (n - d - 1) = 2n - (2d + 1) = n$.

So in all cases, Δ_4 is weakly-Hamiltonian. The proof of Proposition 18 also suggests a relabeling that works: $\ell_1 \stackrel{\text{def}}{=} 1$, $\ell_2 \stackrel{\text{def}}{=} 2$, \dots , $\ell_{d+1} \stackrel{\text{def}}{=} d + 1$, $\ell_{d+2} = n$, $\ell_{d+3} \stackrel{\text{def}}{=} n - 1$, \dots , $\ell_n \stackrel{\text{def}}{=} d + 2$.

To see in what sense Proposition 18 is a higher-dimensional version of Ore's theorem 4, part (A), the best is to give a proof of the latter using the former:

Proof of Ore's theorem 4, part (A). By contradiction, let G be a non-Hamiltonian graph satisfying $\deg u + \deg v \geq n$ for all non-adjacent vertices u, v . Add edges to it until you reach a *maximal* non-Hamiltonian graph G^* . Since any further edge between the existing vertices would create a Hamiltonian cycle, G^* is traceable, and obviously it still satisfies $\deg u + \deg v \geq n$. By Proposition 18 G^* is (weakly-)Hamiltonian, a contradiction. \square

It is possible that the bound of Proposition 18 can be improved. But in any case, the possible improvement could only be small, as the following construction shows.

Non-Example 22. Let $d < m$ be positive integers. Take the disjoint union of two copies A', A'' of Σ_m^d . Let μ be any facet of Σ_m^d and let μ', μ'' be its copies in A' and A'' , respectively. Glue to $A' \cup A''$ a triangulation without interior vertices of the prism $\mu \times [0, 1]$, so that the lower face $\mu \times \{0\}$ is identified with μ' , and the upper face $\mu \times \{1\}$ is identified with μ'' . Let Δ_5 be the resulting d -complex on $n = 2m$ vertices. This Δ_5 is traceable: the added prism, triangulated as a path of d -faces, serves as "bridge" to move between the two copies of Σ_m^d . However, this bridge can only be traveled once, so Δ_5 is not weakly-Hamiltonian. For the labeling that makes it traceable, $d_\sigma + d_\tau = (m - d) + (m - d) = n - 2d$.

Our next result is a "Pósa-type" result, in the sense that it extends most of Nash-Williams' proof [Nas66] of Pósa's theorem [Pós62] to all dimensions. We focus on complexes Δ with the property that any labeling that makes them weakly-traceable, makes them also traceable. Such class is nonempty: for example, it contains all 1-dimensional complexes and all trees of d -simplices (i.e. all triangulations of the d -ball whose dual graph is a tree).

Proposition 23. *Let Δ be any traceable pure d -complex on n vertices, $n > 2d$. Suppose that any labeling that makes Δ weakly-traceable makes it also traceable.*

Let $\sigma_1, \sigma_2, \dots, \sigma_s$ be an ordering of the $(d - 1)$ -faces of Δ^ , such that the respective degrees $d_i \stackrel{\text{def}}{=} d_{\sigma_i}$ are weakly-increasing, $d_1 \leq d_2 \leq \dots \leq d_s$. If for every $d \leq k < \frac{n}{2}$ one has $d_{k-d+1} > k$, then Δ is weakly-Hamiltonian.*

Proof. Among all possible labelings that make Δ weakly-traceable (and thus traceable, by assumption), choose one that maximizes $d_\sigma + d_\tau$, where σ is the $(d - 1)$ -face of H_1 spanned by the first d vertices (that is, $1, 2, \dots, d$) and τ is the $(d - 1)$ -face of H_{n-d} spanned by the last d vertices (that is, $n - d + 1, \dots, n$). Since $n > 2d$, the faces σ and τ are disjoint. If $d_\sigma + d_\tau \geq n$, using the proof of Proposition 18 we get that Δ is weakly-Hamiltonian, and we are done. If not, then one of σ, τ has degree $< \frac{n}{2}$. Up to reversing the labeling, which would swap σ and τ , we can assume that $d_\sigma < \frac{n}{2}$. Now let $J \stackrel{\text{def}}{=} \{d + 2, d + 3, \dots, n - d\}$. For every i in J , which has cardinality $n - 2d - 1$, consider the two d -faces of Σ_n^d

$$S_i \stackrel{\text{def}}{=} \sigma * i \quad \text{and} \quad T_i \stackrel{\text{def}}{=} (i - 1) * \tau.$$

We may assume that at most one of these two faces is in Δ , otherwise a weakly-Hamiltonian cycle arises, exactly as in the proof of Proposition 18. Now for each i in $J_1 \stackrel{\text{def}}{=} \{i \in J : \sigma * i \in \Delta\}$, consider the $(d-1)$ -face ρ_i with vertices $\{i-d, i-d+1, \dots, i-1\}$.

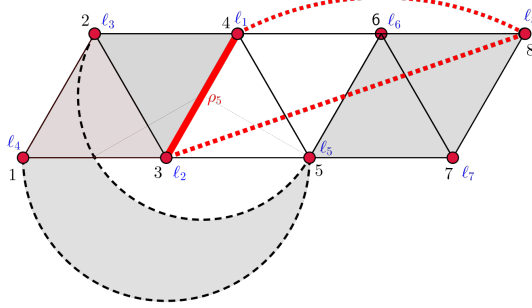


Figure 2: A higher-dimensional “Pósa flip”: Since 125 is in Δ , the vertex 5 is in J_1 . Now the red triangle $\rho_5 * w = 348$ cannot be in Δ , or else we would have a weakly-Hamiltonian cycle with the blue labeling. The blue labeling makes Δ weakly-traceable, with ρ_5 playing the role of the “first” $(d-1)$ -face; the “last” $(d-1)$ -face is the same as before. By how our original labeling was chosen, $d_{\rho_5} \leq d_\sigma < \frac{n}{2}$.

If for some i in J_1 the d -face $\rho_i * w$ is in Δ , then there is a new relabeling ℓ_1, \dots, ℓ_n of the vertices for which we have a weakly-Hamiltonian cycle: see Figure 2 above. (The proof is essentially identical to that of Proposition 18, up to replacing T_i with $T'_i \stackrel{\text{def}}{=} \rho_i * w$, reversing the order, and permuting it cyclically, so that ρ_i is the first face.) So also in this case, we are done.

It remains to discuss the case in which for all $i \in J_1 \stackrel{\text{def}}{=} \{i \in J : \sigma * i \in \Delta\}$, the d -face $\rho_i * w$ is not in Δ . In this case the relabeling ℓ_1, \dots, ℓ_n introduced above makes Δ weakly-traceable, and thus traceable by assumption. For such relabeling, the $(d-1)$ -faces spanned by the first and the last d vertices are ρ_i and τ , respectively. So by the way our original labeling was chosen, $d_{\rho_i} + d_\tau \leq d_\sigma + d_\tau$, and in particular

$$d_{\rho_i} \leq d_\sigma < \frac{n}{2}.$$

Now, any d -face containing σ is of the form $\sigma * v$, where v is either in the set J_1 or in the set $Z \stackrel{\text{def}}{=} \{d+1, n-d+1, \dots, n-1\}$, which has cardinality d . So $d_\sigma \leq |J_1 \cup Z|$. Since J and Z are disjoint, and $J_1 \subset J$, the sets J_1 and Z are also disjoint and we have

$$d_\sigma - d = d_\sigma - |Z| \leq |J_1 \cup Z| - |Z| = |J_1| + |Z| - |Z| = |J_1|.$$

So the set $\{\rho_i : i \in J_1\}$ contains at least $d_\sigma - d$ faces of dimension $d-1$ and degree $\leq d_\sigma$. If we count also σ , we have in Δ at least $d_\sigma - d + 1$ faces of dimension $d-1$ and degree $\leq d_\sigma$. But then, setting $k \stackrel{\text{def}}{=} d_\sigma$, we obtain

$$d_{k-d+1} \leq k < \frac{n}{2},$$

which contradicts the assumption. \square

Again, to see in what sense Proposition 23 is a higher-dimensional version of Pósa’s Theorem 5, perhaps the best is to see how easily the latter follows from the former:

Proof of Pósa’s theorem 5. By contradiction, if G is not Hamiltonian, we can add edges to it until we reach a maximal non-Hamiltonian graph G^* , which still satisfies the degree conditions and is traceable. By Proposition 23, G^* is (weakly-)Hamiltonian, a contradiction. \square

A natural question is whether one can generalize to higher dimensions also part (B) of Ore’s theorem 4. The answer is positive, although some extra work is required. In fact, for graphs part (B) of Ore’s theorem can be quickly derived from part (A) by means of a coning trick. This trick however does not extend to higher dimensions, as we explained in Remark 16, so we’ll have to take a long detour, which makes the proof three times as long. The bored reader may skip directly to the next section.

Definition 24. A d -dimensional complex Δ is *quasi-traceable* if there exists a vertex labeling for which $\Delta \cup H_j$ is weakly-traceable, and moreover, with respect to the same labeling,

- (a) if $j = 1$, then Δ contains all of H_2, \dots, H_{n-d} (i.e., $\Delta \cup H_1$ is traceable);
- (b) if $j \in \{2, \dots, n - 2d\}$, then Δ already contains all of H_1, \dots, H_{j-1} and H_{j+d}, \dots, H_{n-d} (i.e., $\Delta \cup H_j \cup \dots \cup H_{j+d-1}$ is traceable);
- (c) if $j \in \{n - 2d + 1, \dots, n - d - 1\}$, then Δ contains all of H_1, \dots, H_{j-1} and also H_{n-d} (i.e., $\Delta \cup H_j \cup \dots \cup H_{n-d-1}$ is traceable);
- (d) if $j = n - d$, then Δ already contains all of H_1, \dots, H_{n-d-1} (i.e., $\Delta \cup H_{n-d}$ is traceable).

Example 25. The complex $\Delta_6 = 123, 234, 567, 678, 789$ is quasi-traceable, although not weakly-traceable. In fact, Δ_6 becomes weakly-traceable if we add one of the facets 345 and 456, and it becomes even traceable if we add both.

Definition 24 allows the “added faces” to be already present in Δ . In particular, all traceable complexes are quasi-traceable. Here comes our high-dimensional version of Theorem 4, part (B):

Proposition 26. *Let Δ be a quasi-traceable d -dimensional simplicial complex on n vertices, $n > 2d$. If in some labeling that makes Δ quasi-traceable the two $(d - 1)$ -faces σ and τ formed by the first d and the last d vertices satisfy $d_\sigma + d_\tau \geq n - 1$, then Δ is weakly-traceable.*

Proof. By contradiction, suppose Δ is not weakly-traceable; we treat the four cases of Definition 24 separately.

Case (a) is symmetric to Case (d), so we will leave it to the reader.

Case (b) is the main case. Since $j \in \{2, \dots, n - 2d\}$, by definition Δ contains all of H_1, \dots, H_{j-1} and also H_{j+d}, \dots, H_{n-d} . Since Δ is not weakly-traceable, it does not contain H_j . Moreover, $\sigma * (d + j)$ cannot be a facet of Δ , otherwise the two “halfpaths” above would be connected into a weakly-Hamiltonian path. For the same reason, since $(d + j - 1) \in H_{j-1}$, the d -face $(d + j - 1) * \tau$ cannot be in Δ . So let $J' \stackrel{\text{def}}{=} \{d + 2, d + 3, \dots, n - d\} \setminus \{d + j\}$. For every i in J' , which has cardinality $n - 2d - 2$, consider the two d -faces of Σ_n^d

$$S_i \stackrel{\text{def}}{=} \sigma * i \quad \text{and} \quad T_i \stackrel{\text{def}}{=} (i - 1) * \tau.$$

Now there are two subcases: Either there exists an i such that S_i, T_i are both in Δ , or not.

CASE (b.1): *For some i , both S_i and T_i are in Δ .* There are two subsubcases, according to whether i is “before the gap” or “after the gap”.

- **CASE (b.1.1):** $i < d + j$. A weakly-Hamiltonian path arises from a relabeling as follows: We start at the beginning of the second halfpath, with the facets previously called H_{j+d}, H_{j+d+1} , etc., until we reach H_{n-d} . Then we use T_i to get back to the vertex previously labeled by $i - 1$. Next, we use in reverse order the facets previously called $H_{i-d-1}, H_{i-d-2}, \dots, H_2, H_1$. Finally use S_i to jump forward to the vertex previously called i , and conclude the path with the facets previously called $H_i, H_{i+1}, \dots, H_{j-1}$.

- CASE (b.1.2): $i > d + j$. A weakly-Hamiltonian path arises from a relabeling as follows: We start at the beginning of the second halfpath, with the facets previously called H_{j+d}, H_{j+d+1} , etc., until H_{i-d-1} . Then we use T_i to jump forward. As next faces, we use in reverse order the facets previously called $H_{n-d}, H_{n-d-1}, \dots, H_2, H_i$. Finally, we use S_i to jump back to H_1 , and conclude the path with the facets previously called H_1, H_2, \dots, H_{j-1} . So also in this case Δ is weakly-traceable, a contradiction.

CASE (b.2): For all i , at most one of S_i and T_i is in Δ^* . Since the two sets $\{i \in J' : \sigma * i \in \Delta\}$ and $\{i \in J' : (i-1) * \tau \in \Delta\}$ are disjoint, we obtain a numerical contradiction:

$$\begin{aligned} d_\sigma + d_\tau &\leq d + |\{i \in J' : \sigma * i \in \Delta\}| + d + |\{i \in J' : (i-1) * \tau \in \Delta\}| = \\ &= 2d + |\{i \in J' : \sigma * i \in \Delta\} \cup \{i \in J' : (i-1) * \tau \in \Delta\}| \leq \\ &\leq 2d + |J'| = 2d + n - 2d - 2 = n - 2. \end{aligned}$$

Case (c) is the easiest. If $j \in \{n - 2d + 1, \dots, n - d - 1\}$, then H_{j-1} intersects H_{n-d} . Since Δ contains H_1, \dots, H_{j-1} and also H_{n-d} , it is weakly-traceable, a contradiction.

Case (d) is the last one. So, assume $j = 1$ and set $J'' \stackrel{\text{def}}{=} \{d + 2, d + 3, \dots, n - d\}$. For every i in J'' , which has cardinality $n - 2d - 1$, consider the two d -faces of Σ_n^d

$$S_i \stackrel{\text{def}}{=} \sigma * i \quad \text{and} \quad T_i \stackrel{\text{def}}{=} (i - 1) * \tau.$$

Now there are two subcases: Either there exists an i such that S_i, T_i are both in Δ , or not.

CASE (d.1): For some i , both S_i and T_i are in Δ . Then we obtain a weakly-Hamiltonian path as follows: Starting with $\ell_1 = 1$, first we use the face $\sigma * i$, then H_2, \dots, H_{i-d-1} in their order, then we use $(i-1) * n$ to jump forward, and then we come back with H_{n-d}, \dots, H_i .

CASE (d.2): For all i , at most one of S_i and T_i is in Δ . We know by that $\sigma * d = H_1$ is not in Δ because we are treating the case $j = 1$, and we know that $\sigma * n$ is not in Δ otherwise we would have a weakly-Hamiltonian path. Thus any d -face containing σ is of the form $\sigma * v$, where v is either in J'' or in the disjoint set $\{n - d + 1, \dots, n - 1\}$, which has cardinality $d - 1$. In contrast, any d -face containing τ is of the form $(i - 1) * \tau$, where i is either in J'' or in the set $\{2, \dots, d + 1\}$, which has cardinality d . Since the two sets $\{i \in J'' : \sigma * i \in \Delta\}$ and $\{i \in J'' : (i - 1) * \tau \in \Delta\}$ are disjoint, the sum of their cardinality is equal to the cardinality of their union, which is a subset of J'' . So also in this case we obtain a contradiction

$$\begin{aligned} d_\sigma + d_\tau &\leq d - 1 + |\{i \in J'' : \sigma * i \in \Delta\}| + d + |\{i \in J'' : (i - 1) * \tau \in \Delta\}| = \\ &= 2d - 1 + |\{i \in J'' : \sigma * i \in \Delta\} \cup \{i \in J'' : (i - 1) * \tau \in \Delta\}| \leq \quad \square \\ &\leq 2d - 1 + |J''| = 2d - 1 + n - 2d - 1 = n - 2. \end{aligned}$$

Example 27. Let Δ_7 be the simplicial complex on 5 vertices obtained from Σ_5^2 by removing the interior of the two triangles 123 and 124. Clearly Δ_7 is quasi-traceable with $j = 1$, because $\Delta_7 \cup H_1$ is traceable. Since $d_{12} + d_{45} = 4 = n - 1$, by Proposition 26 Δ_7 is weakly-traceable. In fact, the reader may verify that Δ_7 is even Hamiltonian with the relabeling $\ell_1 = 1, \ell_2 = 2, \ell_3 = 5, \ell_4 = 3, \ell_5 = 4$.

For completeness, we conclude this section by showing how Proposition 26 implies part (B) of Ore's theorem 4:

Proof of Ore's theorem 4, part (B). By contradiction, let G be a non-traceable graph satisfying $\deg u + \deg v \geq n - 1$ for all non-adjacent vertices u, v . Add edges to it until we reach a maximal non-traceable graph G^* . This G^* is quasi-traceable and still satisfies $\deg u + \deg v \geq n - 1$. By Proposition 26 G^* is (weakly-)traceable, a contradiction. \square

2 Interval graphs and semiclosed complexes

In the present section,

- (1) we introduce “weakly-closed d -complexes”, generalizing co-comparability graphs;
- (2) we create a hierarchy of properties between closed and weakly-closed complexes, among which a d -dimensional generalization of interval graphs; and
- (3) we connect such hierarchy to traceability and chordality.

2.1 A foreword on interval graphs and related graph classes

INTERVAL GRAPHS are the intersection graphs of intervals of \mathbb{R} . They have long been studied in combinatorics, since the pioneering papers by Lekkerkerker–Boland [LB62] and Gilmore–Hoffman [GH64], and have a tremendous amount of applications; see e.g. [Gol80, Ch. 8, Sec. 4] for a survey. UNIT-INTERVAL GRAPHS, also known as “indifference graphs” [Rob69] or “proper interval graphs”, are the intersection graphs of *unit* intervals, or equivalently, the intersection graphs of sets of intervals no two of which are nested. The claw $K_{1,3}$ is the classical example of a graph that can be realized as intersection of four intervals, three of which contained in the forth; but it cannot be realized as intersection of unit intervals.

Bertossi noticed in 1983 that connected unit-interval graphs are traceable [Ber83], whereas connected interval graphs in general are not: The claw strikes. All 2-connected unit-interval graphs are Hamiltonian [CCC97][PD03]; again, this does not extend to 2-connected interval graphs. That said, for interval graphs (and even co-comparability graphs, see below for the definition) the Hamiltonian Path Problem and the Longest Path Problem can be solved in polynomial time [DS52] [MC12], whereas for arbitrary graphs both problems are well known to be NP-complete, cf. [Kar72].

Given a finite set of intervals in the horizontal real line, we can swipe them “left-to-right”, and thus order them by increasing left endpoint. This so-called “canonical labeling” of the vertices of an interval graph obviously satisfies the following property: for all $a < b < c$,

$$ac \in G \implies ab \in G. \tag{4}$$

This “under-closure” is a characterization: It is easy to prove by induction that any graph with n vertices labeled so that (4) holds can be realized as the intersection graph of n intervals. This result was first discovered by Olario, cf. [LO93, Proposition 4].

There is a “geometrically dual argument” to the one above: Given a finite set of intervals in \mathbb{R} , we could also swipe them right-to-left, thereby ordering the intervals by decreasing right endpoint. This yields a vertex labeling that again satisfies (4), for the same geometric reasons. In general, since some of the intervals may be nested, this “dual labeling” bears no relation with the canonical one. But if we start with a finite set of *unit* intervals, then the dual labeling is simply the reverse of the canonical labeling. Thus in *unit-interval* graphs, not only the canonical labeling is under-closed, but also its reverse is. Or equivalently, in unit-interval graphs, the canonical labeling is closed ‘both below and above’: in mathematical terms, for all $a < b < c$,

$$ac \in G \implies ab, bc \in G. \tag{5}$$

Again, it is not difficult to prove by induction that any graph with n vertices, labeled so that (5) holds, can be realized as the intersection graph of n unit intervals [LO93, Theorem 1]; see Gardi [Gar07] for a computationally-efficient construction.

Recently Herzog et al. [H&10, E&13] rediscovered unit-interval graphs from an algebraic perspective, which will be discussed in the next chapter. They called them *closed graphs* and expanded the notion to higher dimensions as well (“*closed d -complexes*”). Later Matsuda [Mat18]

extended this algebraic approach to the broader class of “co-comparability graphs” (or “weakly-closed graphs”), that we shall now describe in terms of their complement.

Any graph can be given an acyclic orientation by choosing a vertex labeling and then by directing all edges from the smaller to the larger endpoint. Every acyclic orientation can be induced this way. (This is not a bijection: different labelings may induce the same orientation). The DRAWINGS OF POSETS, also called *comparability graphs*, admit also *transitive* orientations, namely, orientations such that if \vec{ab} and \vec{bc} are present, so is \vec{ac} . Let us rephrase this in terms of a vertex labeling, which happens to be the same as a choice of a linear extension of the poset: Comparability graphs are those graphs \overline{G} that admit a labeling such that, for all $a < b < c$,

$$ab \in \overline{G} \text{ and } bc \in \overline{G} \implies ac \in \overline{G}.$$

Not all graphs admit transitive orientations: The pentagon, for example, does not.

CO-COMPARABILITY GRAPHS, also called WEAKLY-CLOSED GRAPHS in [Mat18], are by definition the complements of comparability graphs. So they have a labeling that satisfies the contrapositive of the property above: Namely, for all $a < b < c$,

$$ac \in G \implies ab \in G \text{ or } bc \in G. \tag{6}$$

By comparing (4) and (6), it is clear that all interval graphs are co-comparability.

We should mention other two famous properties that all interval graphs enjoy. A graph is PERFECT if its chromatic number equals the size of the maximum clique. For example, even cycles are perfect, but odd cycles are not, because they have chromatic number 3 and maximal cliques of size 2. Note that in poset drawings, a clique (resp. an independent set) is just a chain (resp. an antichain) in the poset, whereas a coloring represents a partition of the poset into antichains. Thus Dilworth’s theorem (“for every partially ordered set, the maximum size of an antichain equals the minimum number of chains into which the poset can be partitioned” [Dil50] – see Fulkerson [Ful56] for an easy proof) can be equivalently stated as “every co-comparability graph is perfect”. Not all perfect graphs are co-comparability, as shown by large even cycles.

Last property: A graph is CHORDAL if it has no induced subcycles of length ≥ 4 . One can characterize chordality in the same spirit of (4), (5) and (6): Namely, a graph is chordal if and only if it admits a labeling such that, for all $a < b < c$,

$$ac, bc \in G \implies ab \in G. \tag{7}$$

In fact, if a graph G has a labeling that satisfies (7), then G is obviously chordal, because if c is the highest-labeled vertex in any induced cycle, then its neighbors a and b in the cycle must be connected by a chord by (7). The converse, first noticed by Fulkerson–Gross [FG65], follows recursively from Dirac’s Lemma that every chordal graph has a “simplicial vertex”, i.e. a vertex whose neighbors form a clique (cf. [Gol80, p. 83] for a proof). In fact, let us pick any simplicial vertex and label it by n . Then, in the (chordal!) subgraph induced on the unlabeled vertices, let us pick another simplicial vertex and label it by $n - 1$; and so on. The result is a labeling that satisfies (7). See [Gol80, pp. 84–87] for two algorithmic implementations.

Now, if the *same* labeling satisfies (6) & (7), then it trivially satisfies (4); and conversely, if (4) holds, then also (6) & (7) trivially hold. Thus it is natural to conjecture that interval graphs are the same as the *co-comparability chordal* graphs. The conjecture is true, although the ‘obvious’ proof does not work: Some labelings on chordal graphs satisfy (6) but not (4), like 13, 23, 24 on the three-edge path. However, Gilmore–Hoffman proved that any labeling that satisfies (6) on a chordal graph (or more generally, on a graph that lacks induced 4-cycles) can be modified in a way that ‘linearly orders’ all maximal cliques [Gol80, Theorem 8.1] and thus satisfies (4). For more characterizations, and a proof that all chordal graphs are perfect, see Golumbic [Gol80, Chapter 4].

2.2 Higher-dimensional analogs and a hierarchy

A d -dimensional extension* of Characterization (7) of chordality was provided in 2010 by Emtander [Emt10], and is equivalent to the following:

Definition 28 (chordal). Let Δ be a pure d -dimensional simplicial complex with n vertices. Δ is called *chordal* if there exists a labeling $1, \dots, n$ of its vertices (called a “PEO” or “Perfect Elimination Ordering”) such that for any two facets $F = a_0 a_1 \cdots a_d$ and $G = b_0 \cdots b_d$ of Δ with $a_d = b_d$, the complex Δ contains the full d -skeleton of the simplex on the vertex set $F \cup G$.

In 2013, Characterization (5) of unit-interval graphs was generalized as well:

Definition 29 (closed [E&13]). Let Δ be a pure d -dimensional simplicial complex with n vertices. Δ is called *closed* if there exists a labeling $1, \dots, n$ of its vertices such that for any two facets $F = a_0 a_1 \cdots a_d$ and $G = b_0 \cdots b_d$ of Δ with $a_i = b_i$ for some i , the complex Δ contains the full d -skeleton of the simplex on the vertex set $F \cup G$.

Obviously, closed implies chordal. We now present four notions that in the strongly connected case are progressive weakenings of the closed property (see Theorem 50 and Proposition 54 for the proofs); the first property still implies chordality, whereas the last three do not. In Section 2.3, we connect all these notions to traceability (Theorem 63). One of these properties is “new” even for $d = 1$: We will see its importance in Chapter 3.

Definition 30 (unit-interval). Let Δ be a pure d -dimensional simplicial complex with n vertices. The complex Δ is called *unit-interval* if there exists a labeling $1, \dots, n$ of its vertices such that for any d -face $F = a_0 a_1 \cdots a_d$ of Δ , the complex Δ contains the whole d -skeleton of the simplex with vertex set $\{a_0, a_0 + 1, a_0 + 2, \dots, a_d\}$.

Definition 31 (under-closed). Let Δ be a pure d -dimensional simplicial complex with n vertices. The complex Δ is called *under-closed* if there exists a labeling $1, \dots, n$ of its vertices such that for any d -face $F = a_0 a_1 \cdots a_d$ of Δ the following condition holds:

- all faces $a_0 i_1 i_2 \dots i_d$ of Σ_n^d with $i_1 \leq a_1, i_2 \leq a_2, \dots, i_d \leq a_d$, are in Δ .

Definition 32 (semi-closed). Let Δ be a pure d -dimensional simplicial complex with n vertices. The complex Δ is called *semi-closed* if there exists a labeling of its vertices such that for any d -face $F = a_0 a_1 \dots a_d$ of Δ , at least one of the two following conditions holds:

- either all faces $a_0 i_1 i_2 \dots i_d$ of Σ_n^d with $i_1 \leq a_1, i_2 \leq a_2, \dots, i_d \leq a_d$, are in Δ ,
- or all faces $i_0 i_1 \dots i_{d-1} a_d$ of Σ_n^d with $i_0 \geq a_0, i_1 \geq a_1, \dots, i_{d-1} \geq a_{d-1}$ are in Δ .

Definition 33 (weakly-closed). Let Δ be a pure d -dimensional simplicial complex with n vertices. Δ is called *weakly-closed* if there exists a labeling $1, \dots, n$ of its vertices such that for each d -face $F = a_0 a_1 \cdots a_d \in \Delta$, for every integer $g \notin F$ with $a_0 < g < a_d$, there exists a d -face $G = b_0 b_1 \cdots b_d$ in Δ such that G contains g , G is adjacent to F , and at least one of the following two conditions hold:

- either $b_d \neq a_d$,
- or $b_0 \neq a_0$.

Remark 34. For $d = 1$, and assuming connectedness:

- “closed 1-complexes” and “unit-interval 1-complexes” are the same as the *unit interval graphs*; compare Looges–Olario [LO93, Theorem 1] and Matsuda [Mat18, Prop. 1.3].

*Several different d -dimensional generalizations of chordality exist in the literature, e.g. toric chordality [ANS16] or ridge-chordality, cf. e. g. [BB21]. Emtander chose the name “ d -chordal” for what here we call “chordal”.

- “under-closed 1-complexes” are the same as the *interval graphs*, cf. [LO93, Proposition 4].
- “weakly-closed 1-complexes” are the same as the *co-comparability graphs*; this is clear from the definition we gave, but a proof is also in Matsuda [Mat18, Theorem 1.9].

We will see that “semi-closed 1-complexes” are an intermediate class between the previous two. For example, such class contains the 4-cycle but not the complement of long even cycles, as we will prove in Theorem 50.

Remark 35 (“unit-interval” vs. “chordal”). Suppose F and G are two faces of a complex Δ with $\min F = \min G$. Then any of the two conditions “ Δ is closed”, “ Δ is unit-interval” forces Δ to contain the full d -skeleton of the simplex on the vertex set $F \cup G$. (Instead, the condition “ Δ is under-closed” does not suffice: See Remark 36 below). Symmetrically, if F and G are d -faces of Δ with $\max F = \max G$, and Δ is either closed or unit-interval, then Δ must contain the full d -skeleton of the simplex on the vertex set $F \cup G$. For this reason, all unit-interval d -dimensional complexes are chordal.

Remark 36 (“Under-closed” vs. “chordal”). Not all chordal complexes are under-closed: Already for $d = 1$, the chordal graph $G = 12, 13, 14, 23, 25, 36$, known as “3-sun” or “net graph”, is neither interval nor co-comparability. However, while all interval graphs are chordal (and co-comparability), the statement “all under-closed d -complexes are chordal” is false for $d > 1$. In fact, we leave it to the reader to verify that the smallest counterexample is the 2-complex

$$\Delta \stackrel{\text{def}}{=} 123, 124, 234, 235.$$

The other direction in Gillmore–Hoffman’s theorem (namely, “all chordal co-comparability graphs are interval graphs”) does not extend to $d > 1$ either, as the next Proposition shows.

Proposition 37. (i) *Some chordal simplicial complexes are semi-closed, but not under-closed.*
(ii) *If a simplicial complex is chordal and semi-closed with respect to the same labeling, then with respect to that labeling the complex is also under-closed.*

Proof. (i) The example we found is the complex

$$\Sigma = 123, 124, 134, 135, 167, 234, 246.$$

The labeling above is a PEO, so Σ is chordal. A convenient relabeling (we leave it to the reader to figure out the bijection from the vertex degrees) allows us to rewrite it as

$$\Sigma = 123, 256, 345, 346, 347, 356, 456.$$

With this new labeling we see that Σ is weakly- and semi-closed. However, with the help of a software designed by Pavelka [Pav21], we verified that Σ is not under-closed.

- (ii) Let Δ be a simplicial complex with a labeling that is a PEO and makes Δ semi-closed. Let $F = a_0 \cdots a_d$ be a face of Δ with $\text{gap } F > 0$. Let $G = a_0 b_1 \cdots b_d$ be a different d -face of Σ_n^d such that $G \leq F$ (componentwise) and $\min G = \min F$. We claim that for any b_i not in F , there exists a d -face A_i of Σ_n^d that contains b_i , such that $A_i \geq F$ (componentwise) and $\max A_i = \max F$. In fact, by construction $a_0 < b_i \leq b_d \leq a_d$. Since b_i is not in F , there exists a unique $j \in \{0, \dots, d-1\}$ such that $a_j < b_i < a_{j+1}$. Thus if we set

$$A_i \stackrel{\text{def}}{=} a_0 \cdots a_{j-1} b_i a_{j+1} \cdots a_d$$

the claim is proven. Now, either F satisfies condition (i) of the semi-closed definition, and then $G \in \Delta$; or F satisfies condition (ii), in which case all A_i ’s are in Δ . But

by construction, the maximum of all these A_i 's is a_d , the same maximum of F . So by chordality, Δ must contain all the d -faces of Σ_n^d with vertex set contained in

$$F \cup \bigcup_{i \text{ s.t. } b_i \notin F} A_i = \{a_0, a_1, \dots, a_d\} \cup \{b_1, \dots, b_d\} = F \cup G.$$

So also in this case $G \in \Delta$. □

Remark 38. Part (ii) of Proposition 37 is false if one replaces the assumption “semi-closed” with “weakly-closed”: The subcomplex $\Sigma' = 123, 124, 134, 135, 234$ of Σ is weakly-closed and chordal with respect to this labeling, but to prove it under-closed, we need to change labeling.

Remark 39 (“Under-closed” vs. “Shifted”). Recall that a simplicial complex Δ on n vertices is called *shifted* if for every face F of Δ , and for every face G of the simplex on n vertices, if $\dim F = \dim G$ and $F \leq G$ componentwise, then also $G \in \Delta$. Shifted complexes are obviously under-closed. The converse is false, as shown by the graph 12, 23, 34.

Remark 40. Being shifted is maintained under taking cones, by assigning label 1 to the new vertex. In contrast, $G = 12, 13, 23$ is closed and chordal, but the cone over it is neither closed nor chordal. In fact, *none* of the five properties (closed, unit-interval, under-closed, semi-closed, weakly-closed) is maintained under taking cones. A counterexample for all is the unit-interval graph $G = 12, 34, 56, 78$. The cone over G is the U_4^2 of Lemma 43 below.

Let us start exploring the relations between all the new properties with some Lemmas.

Lemma 41. *Let $d \geq k \geq 1$ be integers. If a pure d -dimensional simplicial complex is unit-interval (resp. under-closed, resp. semi-closed, resp. weakly-closed), then its k -skeleton is also unit-interval (resp. under-closed, resp. semi-closed, resp. weakly-closed).*

Proof. It suffices to prove the claim for $k = d - 1$; the general claim follows then by iterating. We prove only the weakly-closed case; the others are easier. Let Δ be a pure weakly-closed d -complex. Let $\sigma = a_0 \cdots a_{d-1}$ be a $(d - 1)$ -face of Δ . Let $g \notin \sigma$ be an integer such that $a_0 < g < a_{d-1}$. Since Δ is pure, there exists a d -face F of Δ that contains σ . Let v be the vertex of F not in σ . If $v = g$, i.e. if $F = \{g\} \cup \sigma$, then all the d facets of Δ different than σ are adjacent to σ and contain g ; if we choose one of these d facets that has either different minimum or different maximum than σ , we are done. So let us assume that $v \neq g$, or equivalently, that F does not contain g . By the weakly-closed assumption, there exists a d -face G in Δ such that G contains g , G is adjacent to F , and G and F do not have same minimum and maximum. If G contains the entire face σ , i.e. $G = \sigma \cup g$, then again we could conclude as above, choosing some facet of G different than σ . So we can assume that G does not contain the whole of σ , or in other words, that the vertex v is present in G . Let τ be the unique face of G that does not contain v . By construction, σ and τ are adjacent, and $g \in \tau$. If σ and τ had same minimum and maximum, then also F and G would, because F and G are obtained by adding to σ and τ , respectively, the same element v . Hence, the $(d - 1)$ -skeleton of Δ is weakly-closed. □

Lemma 42. *Let $d \geq 2$. Let C^{d+1} be the $(d + 1)$ -dimensional simplicial complex with facets H_1 and H_2 . The boundary S^d of C^{d+1} is strongly-connected, semi-closed, but not under-closed. The d -skeleton B^d of C^{d+1} is traceable, strongly-connected, unit-interval, but not closed. In particular, the k -skeleton of a closed complex need not be closed.*

Proof. Note that S^d is B^d minus a d -face, so since $d \geq 2$ the 1-skeleta of B^d and of S^d coincide. The vertices of B^d (respectively, of S^d) can be partitioned with respect to the number of edges

containing them, as follows: exactly two vertices have degree $d + 1$, and we shall call them “apices”; the remaining $d + 1$ have degree $d + 2$, and we shall call them “basepoints”. The crucial remark is that in B^d (resp. S^d) the two apices are not connected by any edge. We claim that any labeling that makes B^d or S^d closed *must* assign labels 1 and $d + 3$ to the two apices. In fact:

- If the label 1 is assigned to a basepoint, let b_1, \dots, b_d be the other d basepoints and let v, w be the apices, with $v < w$. Then B^d (resp. S^d) contains a d -face F of vertices $\{1, b_1, \dots, b_{d-1}, v\}$ and a d -face G of vertices $\{1, b_1, \dots, b_{d-1}, w\}$. Note that 1 is in the same position in F and G , yet B^d (or S^d) does not contain the whole d -skeleton of the simplex on $F \cup G$, because vw is missing. So the closed condition is not satisfied.
- Symmetrically, if $d + 3$ is assigned to a basepoint, call b_1, \dots, b_d the other basepoints and v, w the apices, with $v < w$. Then B^d (resp. S^d) contains a d -face F of vertices $\{v, b_1, \dots, b_{d-1}, d + 3\}$ and a d -face G of vertices $\{w, b_1, \dots, b_{d-1}, d + 3\}$. So $d + 3$ is the maximum of both faces, and again B^d (resp. S^d) does not contain the edge vw , so the closed condition is not met.

Next, we claim that any labeling that makes S^d under-closed *must* assign labels 1 and $d + 3$ to the two apices. (Caveat: This claim is valid only for S^d , since already B^2 is under-closed with the labeling 123, 124, 134, 234, 125, 135, 235, where the apices are 4 and 5.) In fact:

- If the label 1 is assigned to a basepoint, then any other vertex is contained in a facet that contains also 1. The same is true if $d + 3$ is assigned to a basepoint. So either way, there is a face H containing both 1 and $d + 3$. Thus $\text{gap } H = 2$. But then if the labeling is under-closed, the complex must contain all three facets $12 \cdots d j$, with $j \in \{d + 1, d + 2, d + 3\}$. So we found in S^d three different facets containing the $(d - 1)$ -face $\sigma \stackrel{\text{def}}{=} 12 \cdots d$. This is a contradiction because S^d is topologically a sphere: Every $(d - 1)$ -face in it lies in exactly two d -faces.

Thus the two claims are proven. So up to a rotation that does not affect the list of facets, both for B^d and S^d we may focus on the labeling that we introduced from the start. With respect to that labeling, S^d is clearly semi-closed, but it is not under-closed, because the d -face with vertices $2, 3, \dots, d + 1, d + 2$ is missing. Similarly, with respect to that labeling, B^d is traceable and unit-interval, but it is not closed for the following reason. Let F (resp. G) be the face of vertices $1, 3, 4, \dots, d + 1, d + 2$ (resp. $2, 3, 4, \dots, d + 1, d + 3$). Since F (resp. G) is contained in the facet H_1 (resp. H_2) of C^{d+1} , it is in B^d . Yet vertex 3 appears in second position in both F and G . However, the face H_3 of vertices $1, 3, 4, \dots, d + 1, d + 3$ contains the edge connecting the two apices, so H_3 is not in B^d . \square

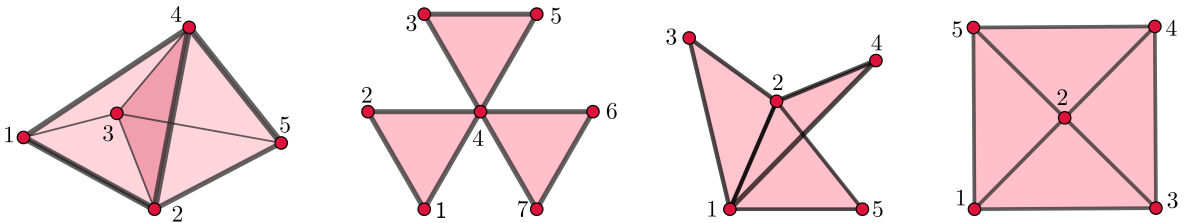


Figure 3: (i) A 2-complex $B^2 = 123, 124, 134, 234, 235, 245, 345$ that is unit-interval, but not closed; if we remove the triangle 234, we get a 2-complex S^2 that is semi-closed, not under-closed, cf. Lemma 42. (ii) A 2-complex $U_3^2 = 124, 345, 467$ that is closed, but not weakly-closed, cf. Lemma 43. (iii) A 2-complex $\Delta_3^2 = 123, 124, 125$ that is under-closed, but not unit-interval, cf. Lemma 44. (iv) A 2-complex $Q^2 = 123, 125, 234, 245$ that is weakly-closed, but not semi-closed, cf. Lemma 46.

Lemma 43. *Let d and k be positive integers. Let U_k^d be a one-point union of k copies of Σ^d . Then U_k^d is closed if and only if $k \leq d + 1$, and it is weakly-closed if and only if $k \leq 2$. In particular, for all $d \geq 2$, the d -complex U_{d+1}^d is closed, but not weakly-closed.*

Proof. Let v be the vertex common to all facets. When $k > d + 1$, by the pigeonhole principle there are two facets in which v appears in the same position; were U_k^d closed, its dual graph would have to contain a clique, which is not the case. When $k \leq d + 1$, we force the closed property by giving v a label so that v appears in a different position in all facets. We show an algorithm to do this in case $k = d + 1$, leaving the case $k < d + 1$ to the reader. We label v by $f_d \stackrel{\text{def}}{=} \binom{d+1}{2} + 1$. We label the vertices of the first facet by $123 \cdots d f_d$: so in the first facet, v comes last. Then for all $i = 2, 3, \dots, k = d + 1$, we label the i -th facet by using the next available $d - i$ integers below f_d , then f_d , then the first $i - 1$ available integers after f_d . This way in the i -th facet, v comes “ i -th last”. For example, the labeling we construct for U_4^3 , since $f_3 \stackrel{\text{def}}{=} \binom{4}{2} + 1 = 7$, is $U_4^3 = 1237, 4578, 67910, 7111213$.

Finally, suppose that U_k^d is weakly-closed. No face of U_k^d has an adjacent facet. Hence, the labeling satisfying the weakly-closed condition must consist only of gap-0 faces. But labeling all facets with consecutive vertices is possible if and only if $k \leq 2$. \square

Lemma 44. *Let $k \geq 1$ and $d \geq 2$ be integers. Let Δ_k^d be the d -dimensional complex on $d + k$ vertices obtained by joining the $(d - 1)$ -simplex Σ^{d-1} to a 0-complex consisting of k points. Then*

- (a) Δ_k^d is under-closed for all k .
- (b) Δ_k^d is closed, if and only if it is unit-interval, if and only if it is (weakly) traceable, if and only if $k \leq 2$.

Proof. Let us label the vertices of Σ^{d-1} by $1, 2, \dots, d$. This labeling immediately shows that Δ_k^d is under-closed. Moreover, the d -complex Δ_k^d is strongly-connected. It has exactly $d + k$ vertices and k facets. When $k \leq 2$ its dual graph is a path, so clearly the obvious, consecutive labeling makes Δ_k^d a closed, unit-interval, and traceable complex. But when $k \geq 3$, the “path of k d -simplices” is not a subcomplex of Δ_d . Hence, for $k \geq 3$ the complex Δ_k^d is not traceable, not weakly-traceable, and not weakly-Hamiltonian. The fact that Δ_k^d is neither unit-interval nor closed can be verified either directly, or using Proposition 54 and Theorem 56 below. \square

Remark 45. The 1-skeleton of $\Delta_3^2 = 123, 124, 125$ (cf. Figure 3) is the graph

$$G_5 = 12, 13, 14, 15, 23, 24, 25$$

which is under-closed by Lemma 41. It is not difficult to see that G_5 is the smallest 2-connected interval graph that is not Hamiltonian.

Lemma 46. *Let $d \geq 2$ be an integer. Let Q^d be the d -dimensional complex on $d + 3$ vertices obtained by taking $d - 1$ consecutive cones over the square. Then Q^d is weakly-closed, but not semi-closed.*

Proof. Both $Q^2 = 123, 125, 234, 245$ and $Q^3 = 1236, 1256, 2346, 2456$ are weakly-closed. If we label further coning vertices using consecutive labels after 6, we claim that the weakly-closed property is maintained. (This is not obvious, as the weakly-closed property is not maintained under arbitrary cones, cf. Remark 40.) In fact, since every face F of Q^3 contains 6, the gap of F equals the gap of $F \cup \{7\}$, and the missing integers are the same, so the calculations proving weakly-closedness end up being the same for Q^3 and Q^4 . For the same reasons, one can show that if some Q^d is semi-closed with a labeling that assigns consecutive labels to two apices, then Q^{d-1} is semi-closed too. But if $d \geq 7$, Q^d has ≥ 10 vertices, and only 4 of them are not apices; so necessarily two apices are assigned consecutive labels. So to complete the proof we only need to show that Q^2, Q^3, Q^4, Q^5 and Q^6 are not semi-closed, which can be verified with [Pav21]. \square

Lemma 47. *Let Δ be a pure d -complex where every vertex is in at most k facets.*

(1) *In any labeling that makes Δ weakly-closed, every facet has $\text{gap} \leq 2k - 2$.*

(2) *In any labeling that makes Δ semi-closed, every facet has $\text{gap} \leq k - 1$.*

If in addition $d = 1$ and Δ is a k -regular graph, then in any labeling that makes Δ semi-closed, the k edges of the type $1j$, with $2 \leq j \leq k + 1$, are all in Δ ; and so are all the k edges of the type in , with $n - k \leq i \leq n - 1$.

(3) *In any labeling that makes Δ unit-interval, every facet has $\text{gap} \leq g$, where g is the largest integer such that $\binom{g+d}{d} \leq k$; in particular, every facet has $\text{gap} \leq \sqrt[d]{kd!} - 1$.*

Proof. For any vertex v of Δ , let $\deg v$ be the number of facets of Δ containing it. For any facet F of Δ , let S_F be the set of integers $i \notin F$ such that $\min F < i < \max F$. By definition, S_F has cardinality equal to $\text{gap } F$. For brevity, set $a \stackrel{\text{def}}{=} \min F$ and $b \stackrel{\text{def}}{=} \max F$.

(1) For every i in S_F , there is a face G_i adjacent to F that contains the vertex i and exactly d vertices of F , among which exactly one of a, b . Clearly as i ranges over S_F , the G_i 's are all different. So $\deg a + \deg b \geq \text{gap } F + 2$. (The summand 2 is due to the fact that we should count also F itself, once contributing to $\deg a$ and once to $\deg b$). Since $k \geq \deg a$ and $k \geq \deg b$, we conclude that $\text{gap } F \leq 2k - 2$.

(2) For every i in S_F , either Δ contains the $n_a \geq \text{gap } F + 1$ facets (including F itself) with minimum a that are componentwise $\leq F$, or Δ contains the $n_b \geq \text{gap } F + 1$ facets (including F itself) with maximum b that are componentwise $\geq F$. Either way, there is a vertex v (either a or b) with $\deg v \geq \text{gap } F + 1$. Since $\deg v \leq k$ by assumption, we conclude that $\text{gap } F \leq k - 1$. So the first claim is settled. From this applied to $d = 1$, it follows that

$$\{ \text{edges of } \Delta \text{ containing } 1 \} \subseteq \{ 1j \text{ such that } 2 \leq j \leq k + 1 \}.$$

The two sets above have size $\deg 1$ and k , respectively. If Δ is k -regular, the two quantities are equal, hence the sets coincide. The same argument applies to the edges containing n .

(3) For every i in S_F , by definition of unit-interval, Δ contains the $\binom{\text{gap } F + d}{d}$ d -faces that contain vertex i and have vertices in $\{a, a + 1, \dots, b\}$. So we must have $\binom{\text{gap } F + d}{d} \leq k$. In particular, since $\binom{g+d}{d} \geq \frac{(g+1)^d}{d!}$ for all positive integers g, d , we cannot have $\frac{(\text{gap } F + 1)^d}{d!} > k$. \square

Our next Lemma is a d -dimensional version of the well-known fact that cycles of length 5 or more are not co-comparability, cf. Matsuda [Mat18].

Lemma 48. *For $n \geq 2d + 3$, the d -dimensional annulus $A_n^d \stackrel{\text{def}}{=} H_1, H_2, \dots, H_n$ and any k -skeleton of it are not weakly-closed.*

Proof. By Lemma 41, it suffices to prove that the 1-skeleton G of A_n^d is not weakly-closed. By contradiction, let a_1, \dots, a_n be a re-labeling of the vertices $1, \dots, n$ (respectively) that proves G weakly-closed. Up to rotating the labeling cyclically, we can assume that a_1 is the smallest of the a_i 's. Since $n \geq 2d + 3$, in particular $n - d > d + 2$, so the labels $a_{n-d}, a_{n-d+1}, \dots, a_n, a_1, a_2, \dots, a_{d+2}$ are all distinct. Were $a_{d+2} < a_n$, we would have a contradiction with the weakly-closed assumption: $a_1 a_n$ is in G , but neither $a_1 a_{d+2}$ nor $a_{d+2} a_n$ is. So $a_n < a_{d+2}$. Symmetrically, were $a_{d+1} > a_{n-d}$, we would have a contradiction: $a_1 a_{d+1}$ is in G , but neither $a_1 a_{n-d}$ nor $a_{n-d} a_{d+1}$ is. So $a_{d+1} < a_{n-d}$. Now let us compare a_{d+1} and a_n :

- If $a_{d+1} > a_n$, then $a_n < a_{d+1} < a_{n-d}$ by what we said above; so we get a contradiction, because the edge $a_n a_{n-d}$ is in G , but neither $a_n a_{d+1}$ nor $a_{d+1} a_{n-d}$ is.
- If $a_{d+1} < a_n$, then $a_{d+1} < a_n < a_{d+2}$ by what we said above; so symmetrically we get another contradiction, because $a_{d+1} a_{d+2}$ is in G , but neither $a_{d+1} a_n$ nor $a_n a_{d+2}$ is. \square

Remark 49. A_6^2 is not weakly-closed, even if its 1-skeleton is semi-closed [Pav21]. (A_5^2 instead is weakly-closed.) So the bound $n \geq 2d + 3$ of Lemma 48 is best possible in general, but if one only cares about A_n^d and not about its skeleta, then it can be improved.

Theorem 50. For each $d \geq 1$, for (pure) simplicial d -complexes, one has the hierarchy

$$\{ \text{unit-interval} \} \subsetneq \{ \text{under-closed} \} \subsetneq \{ \text{semi-closed} \} \subsetneq \{ \text{weakly-closed} \} \subsetneq \{ \text{all} \}.$$

Proof. All inclusions are obvious except perhaps the third one. Let $F = a_0 a_1 \dots a_d$ be a face of Δ . If F satisfies condition (i) in the definition of semi-closed, and there is a g such that $a_i < g < a_{i+1}$, then $G' \stackrel{\text{def}}{=} a_0 a_1 \dots a_i g a_{i+1} \dots a_d$ is componentwise $\leq F$ and thus belongs to Δ ; moreover, since $\max G' < \max F$, the face G' satisfies condition (i) in the definition of weakly-closed. If instead F satisfies condition (ii) in the definition of semi-closed, and $a_i < g < a_{i+1}$ for some g , then $G'' \stackrel{\text{def}}{=} a_1 \dots a_i g a_{i+1} \dots a_d$ is componentwise $\geq F$, so G'' is in Δ ; and since $\min G'' > \min F$, this G'' satisfies condition (ii) in the definition of weakly-closed.

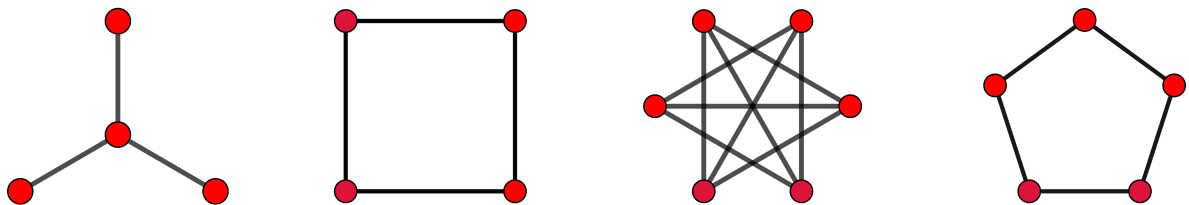


Figure 4: One-dimensional simplicial complexes that are: (i) Not unit-interval, but under-closed. (ii) Not under-closed, but semi-closed. (iii) Not semi-closed, but weakly-closed. (iv) Not even weakly-closed.

Next, we discuss the strictness of the inclusions, which is the interesting part of the theorem.

- (i) For $d = 1$, the claw graph 12, 13, 14 is under-closed only with this labeling, which is not unit-interval because for example 23 is missing.
For $d \geq 2$, strictness follows by Lemma 44.
- (ii) For $d = 1$, the 4-cycle is semi-closed with the labeling 12, 13, 24, 34. By Lemma 47, part (2), *only* this labeling makes the 4-cycle semi-closed. This labeling is not under-closed, because 24 is an edge, but 23 is not. More generally, for any $n \geq 4$, one can show that the graph $\text{susp}(A_{n-2})$ of Remark 12 is semi-closed (with the suspension apices labeled by 1 and n), but not under-closed.
For $d \geq 2$, the strictness of the inclusion follows by Lemma 42.
- (iii) For $d = 1$: Since C_{2k} is a comparability graph (it is the nonempty-face poset of the k -gon), $\overline{C_{2k}}$ is co-comparability. We claim that $\overline{C_{2k}}$ is not semi-closed for any $k \geq 3$. For notational simplicity, we give the proof for $k = 3$; the case of arbitrary k has a completely analogous proof. Suppose by contradiction that $\overline{C_6}$ has a semi-closed labeling. Since C_6 is 2-regular, its complement is $(6 - 1 - 2)$ -regular, i.e. 3-regular. By Lemma 47, part (2), all of 12, 13, 14 and 36, 46, 56 are edges. In contrast, 15, 16 and 26 are not edges, again by Lemma 47. But then 25 must be an edge of $\overline{C_6}$, for otherwise 15, 16, 26 and 25 would form a 4-cycle inside the complement, which is C_6 . We claim that this edge 25 cannot satisfy the semi-closed condition. In fact, if all of 23, 24, 25 were edges, together with 12 we would have 4 edges containing vertex 2, contradicting 3-regularity; and similarly, if all of 25, 35, 45 were edges, counting also 56 we would have 4 edges containing vertex 5. This shows strictness of the inclusion for $d = 1$; the case $d \geq 2$ is settled by Lemma 46.
- (iv) For any $d \geq 1$, this is settled by Lemma 48. \square

2.3 Shortest dual paths and relation with traceability

As we saw in Lemma 43, there exist complexes like $U_3^2 = 124, 345, 467$ that are closed but not weakly-closed. So at this point we owe the reader some explanation: Why did we (and before us, Matsuda [Mat18] and others, in the 1-dimensional case) choose to call “weakly-closed” a property not implied by “closed”? Here is the reason. We are going to show that *all strongly-connected closed complexes are unit-interval* (Proposition 54), so in particular under-closed, semi-closed, and weakly-closed. We will then prove that all such complexes are traceable (Theorem 56), which can be viewed as a higher-dimensional generalization of the graph-theoretical results by Bertossi [Ber83] and Herzog et al’s [H&10, Proposition 1.4]. The key to our generalization is to focus on shortest paths in the *dual* graph.

Definition 51. Let F be a facet a pure d -dimensional simplicial complex Δ . Let v be a vertex of Δ . A *shortest path between F and v* is a path in the dual graph of Δ of minimal length from F to some facet containing v . The *distance* between F and v is the length of a shortest path, if any exists, or $+\infty$, otherwise.

Definition 52. Let Δ be a pure d -dimensional simplicial complex, with vertices labeled from 1 to n . A path F_0, F_1, \dots, F_ℓ in the dual graph of Δ is called *ascending*, if each F_i is obtained from F_{i-1} by replacing the smallest vertex of F_{i-1} , with a vertex greater than all remaining vertices of F_{i-1} . A path is called *descending*, if the reverse path is ascending.

For example, suppose that a 2-complex Δ contains the facets 124, 245, 456, and 356. The dual path they form is not ascending – or better, it is ascending, except for the last step. Such dual path demonstrates that the vertex $v = 3$ is at distance ≤ 3 from 124. Now suppose that we know in advance that Δ is closed: Then from $356, 456 \in \Delta$, we immediately derive that Δ must contain the whole 2-skeleton of the simplex 3456. Note that the same conclusion could be reached also if we knew in advance that Δ is *unit-interval*, rather than closed. Either way: Δ contains the facet $G = 345$ which contains 3 and is adjacent to 245. So 124, 245, 345 yields a “shortcut” to the original path, thereby proving that $v = 3$ is actually at distance ≤ 2 from 124. And it gets even better: Since 245 and 345 are in Δ , by the closed assumption (or the unit-interval assumption) on Δ , we may conclude that Δ contains the whole 2-skeleton of the simplex 2345. So also 234 is in Δ , which means that $v = 3$ is at distance 1 from 124.

This example generalizes as follows, in what can be viewed as a higher-dimensional version of Cox–Erskine’s narrowness property [CE15]:

Lemma 53. *Let Δ be a pure d -dimensional simplicial complex, with a labeling that makes it either closed or unit-interval. Let $F = a_0 a_1 \dots a_d$ be a facet of Δ . Let v be a vertex. If the distance between F and v is a finite number $\ell \geq 2$, then*

- *either there is a shortest path from F to v that is ascending (and thus $v > a_d$),*
- *or there is a shortest path from F to v that is descending (and thus $v < a_0$).*

If instead $a_0 < v < a_d$, and some facet containing v is in the same strongly-connected component of F , then the distance between F and v is at most one, and Δ contains the whole d -skeleton of the simplex on the vertex set $F \cup \{v\}$.

Proof. Let

$$F = F_0, \dots, F_{i-1}, F_i, F_{i+1}$$

be a shortest path from F to a vertex $v \in F_{i+1}$. Suppose the path is ascending until F_i , but it stops being ascending when passing from F_i to F_{i+1} . This means that $\max F_i = \max F_{i+1}$. By Remark 35, Δ contains the whole d -skeleton of the simplex with vertex set $F_i \cup F_{i+1}$. In

particular, if we set $\gamma \stackrel{\text{def}}{=} F_{i-1} \cap F_i$, the complex Δ contains $G \stackrel{\text{def}}{=} \gamma \cup v$. But since G is a d -face that contains v and is already adjacent to F_{i-1} ,

$$F = F_0, \dots, F_{i-1}, G$$

is a shorter path from F to v than the one we started with, a contradiction. The same argument applies to descending paths. If instead $a_0 < v < a_d$, clearly there cannot be any ascending or descending path from F to v . So either $v \in F$, in which case the distance from F to v is 0 and there is nothing to prove, or $v \notin F$, in which case the distance is 1. In the latter case, F and the adjacent face G containing v have same maximum, so again by Remark 35 the complex Δ contains the d -skeleton of the simplex on $F \cup G = F \cup \{v\}$. \square

Proposition 54. *All strongly-connected closed simplicial complexes are unit-interval.*

Proof. Let Δ be a strongly-connected d -dimensional simplicial complex that is closed with respect to some-labeling. Let $F = a_0 a_1 \cdots a_d \in \Delta$. We claim the following:

(*) If there exist $m \in \{1, \dots, d\}$ and g_1, \dots, g_m not in F , with $a_0 < g_1 < g_2 < \dots < g_m < a_d$, then Δ contains the d -skeleton of the simplex with vertex set $\{a_0, \dots, a_d, g_1, \dots, g_m\}$.

If $\text{gap}(F) = 0$, then the implication is trivially true, because the antecedent is never verified. So suppose $\text{gap}(F) > 0$, and let us proceed by induction on m .

For $m = 1$: Pick a vertex g of Δ not in F , with $a_0 < g < a_d$. Since Δ is strongly connected, by the second part of Lemma 53 the complex Δ has a facet G that contains g and is adjacent to F . Had G neither same minimum nor same maximum of F , then either $G = a_1 a_2 \cdots a_d g$ or $G = g a_0 a_1 \cdots a_{d-1}$. But both cases contradict the assumption $a_0 < g < a_d$. Hence, F and G have either same minimum or same maximum (or both), so they share at least one vertex in the same position. Since Δ is closed, Δ contains the d -skeleton of the simplex on $F \cup G = F \cup \{g\}$.

For $m > 1$: let H be a subset of $\{a_0, \dots, a_d, g_1, \dots, g_m\}$ of cardinality $d + 1$. If H contains at most $m - 1$ elements of $\{g_1, \dots, g_m\}$, then we know that $H \in \Delta$ by the inductive assumption. If g_1, \dots, g_m are all vertices of H , let us consider a new face H' with exactly the same vertices of H , except for one replacement, to be decided as follows:

- If $\min H = a_0$ and $\max H = a_d$, we shall replace g_1 with any vertex v of F that is not in H . This way, since $a_0 \leq v \leq a_d$, we have that as real intervals

$$(\min H, \max H) = (a_0, a_d) = (\min H', \max H').$$

- If $\min H = g_1$, or if $\min H = a_i$ for some $i > 0$, we shall replace g_1 with a_0 . This way

$$(\min H, \max H) \subsetneq (a_0, \max H) = (\min H', \max H').$$

- If $\max H = g_m$, or $\max H = a_i$ for some $i < d$, we shall replace g_m with a_d . This way

$$(\min H, \max H) \subsetneq (\min H, a_d) = (\min H', \max H').$$

In all three cases, if w is the only element that belongs to H but not to H' , then w is either g_1 or g_m , and we have

$$\min H' < w < \max H'.$$

Moreover, H' contains at most $m - 1$ elements of $\{g_1, \dots, g_m\}$, so by the inductive assumption H' is in Δ . But since $\min H' < w < \max H'$, by the second part of Lemma 53 we conclude that also H is in Δ . By the genericity of H , this proves Claim (*). From the Claim the conclusion follows immediately, by choosing m maximal. \square

Remark 55. The converse is false: The complex with k disjoint d -simplices is obviously not strongly-connected, yet it is unit-interval with the natural labeling below:

$$\Delta = H_1, H_{d+2}, H_{2d+3}, \dots, H_{(k-1)d+k}.$$

For connected graphs, it is obvious that “closed” and “unit-interval” are the same: This is noticed also in Matsuda [Mat18, Proposition 1.3] and Crupi–Rinaldo [CR14]. However, as we saw in Lemma 42, higher-dimensional complexes that are both strongly-connected and unit-interval might not be closed.

We have arrived to the main result of this section, the generalization of Bertossi’s theorem:

Theorem 56 (Higher-dimensional Bertossi). *Let Δ be a pure d -dimensional simplicial complex that is either closed or unit-interval. Then*

$$\Delta \text{ is strongly-connected} \iff \Delta \text{ is traceable.}$$

Proof.

- \Leftarrow : Let F be a d -face of Δ . We want to find a walk from Δ to H_1 in the dual graph. If $\text{gap } F = 0$, then $F = H_j$ for some j , and H_1, H_2, \dots, H_j is the desired path. If $\text{gap } F > 0$, let $i \stackrel{\text{def}}{=} \min F$. Since F and H_i have same minimum, by Remark 35 Δ contains the whole d -skeleton of the simplex on $F \cup H_i$. But the d -skeleton of a higher-dimensional simplex is strongly-connected, which means that in the dual graph of Δ we can walk from F to H_i . And since H_i has gap 0, we can walk from it to H_1 .
- \Rightarrow : Fix a labeling for which Δ is (almost-)closed. We are going to show by induction on j that with the *same* labeling, every H_j is in Δ . For $j = 1$, since Δ is pure, it contains a face $F = a_0 a_1 \cdots a_d$ with $a_0 = 1$, and then it is easy to derive (either directly, or using that the labeling satisfies the under-closed condition by Theorem 50) that H_1 is in Δ . Now suppose that Δ contains H_j and let us show that Δ contains H_{j+1} . By Lemma 53, Δ has a d -face H' that contains $d + j + 1$ and is adjacent to H_j . Such H' has the same vertices of H_j , with the exception of a single vertex i that was replaced by $d + j + 1$. Now either $i = j$, in which case $H' = H_{j+1}$ and we are done; or $i > j$. If $i > j$, then j was not replaced, so it is still present in H' . Hence H' and H_j are adjacent faces with the same minimum, namely, j . By Remark 35, this implies that H_{j+1} is in Δ . \square

Remark 57. If the “unit-interval” assumption is weakened to “under-closed”, then the direction “ \Rightarrow ” of Theorem 56 no longer holds, with $K_{1,3}$ playing the usual role of the counterexample. The direction “ \Leftarrow ” instead is still valid. We claim in fact that *all weakly-closed traceable complexes are strongly-connected*. To see this, it suffices to show that from any d -face F of positive gap we can walk in the dual graph to some gap-0 face. But the weakly-closed definition tells us how to move in the dual graph from F to a face F' of smaller gap than F . So if we iterate this, eventually we get from F to a gap-0 face. (The same type of argument is carried out in details in the proof of Theorem 63, item (5), below.) That said, the “weakly-closed” assumption is needed for “ \Leftarrow ”. In fact, for any $d \geq 2$, if $G_d \stackrel{\text{def}}{=} \{1, d + 2, 2d + 3, \dots, (k - 1)d + k, kd + (k + 1), \dots, d^2 + d + 1\}$, then the traceable d -complex with $d^2 + d + 1$ vertices $\Delta = H_1, H_2, \dots, H_{d^2}, H_{d^2+1}, G_d$ is not strongly-connected. Its dual graph is a path of length $d^2 + 1$ plus an isolated vertex.

Generalizing a result by Chen, Chang, and Chang [CCC97, Theorem 2], we can push Theorem 56 a bit further. If D is a simplicial complex obtained from Δ by deleting some vertices v_1, \dots, v_k , then any labeling of Δ naturally induces a *compressed labeling* for D , just by ordering the vertices of D in the same way as they are ordered inside Δ . For example, if $\Delta = 123, 134, 345$, the compressed labeling for $D = \text{del}(2, \Delta)$ is 123, 234. A priori, this D need not be pure.

Lemma 58. *Let Δ' be a d -dimensional simplicial complex obtained by deleting some vertices from a d -dimensional simplicial complex Δ . If Δ is unit-interval (resp. under-closed, resp. semi-closed), then so is Δ' .*

Proof. If the original labeling satisfied the unit-interval (resp. under-closed, resp. semi-closed) condition, so does the compressed labeling. \square

Lemma 59. *Let Δ be a d -dimensional strongly-connected simplicial complex, with a labeling that makes it unit-interval. The following are equivalent:*

- (a) *The deletion of d or less vertices, however chosen, yields a d -complex that is strongly connected.*
- (b) *The deletion of d or less vertices, however chosen, yields a pure d -complex that with the compressed labeling is traceable.*
- (c) *Δ contains all faces of gap $\leq d$.*

Proof. (a) \Leftrightarrow (b): By Lemma 58 the compressed labeling satisfies the unit-interval condition. Via Theorem 56, we conclude.

(b) \Rightarrow (c): By deleting zero vertices we notice that Δ is itself traceable. Let $F = a_0 \cdots a_d$ be any d -face of Σ_n^d that has gap $\leq d$. If $\text{gap}(F) = 0$, then F is one of H_1, \dots, H_{n-d} , so F is in Δ by definition of traceable. Otherwise, set $S_F \stackrel{\text{def}}{=} \{j \notin F \text{ such that } a_0 < j < a_d\}$. Let Δ' be the complex obtained from Δ by deleting the vertices in S_F , which are at most d . By assumption, Δ' is traceable with the ‘‘compressed labeling’’. So Δ' contains a gap-0 face of minimum a_0 . But by how the compressed labeling is defined, this face has exactly the vertices that in the original labeling for Δ were called a_0, a_1, \dots, a_d . So F is in Δ .

(c) \Rightarrow (b): Let Δ' be the d -complex resulting from the deletion. With the compressed labeling, Δ' is traceable, because any gap-0 d -face of Δ' with the compressed labeling, is a d -face of Δ that had gap $\leq d$ in the original labeling. It remains to see that Δ' is pure. We prove that Δ' has no facets of dimension $d - 1$, leaving the case of facets of even lower dimensions to the reader. We claim that every $(d - 1)$ -face σ of Δ lies in at least $d + 1$ distinct d -faces of Δ . From the claim the conclusion follows via the pigeonhole principle: If we delete d vertices, however chosen, then at least one of the d -faces containing σ will survive the deletion, which implies that σ is not a facet in Δ' .

So let us prove the claim. Let $\sigma = b_0 \cdots b_{d-1}$. If $b_{d-1} - b_0 - d + 1 \stackrel{\text{def}}{=} \text{gap}(\sigma) \leq d$, then $b_{d-1} + 1 \leq b_0 + 2d$. So for each i in the $(d + 1)$ -element set

$$T_\sigma \stackrel{\text{def}}{=} \{b_0, b_0 + 1, \dots, b_{d-1}, b_{d-1} + 1, \dots, b_0 + 2d\} \setminus \{b_0, b_1, \dots, b_{d-1}\}$$

the d -face $\sigma \cup \{i\}$ has gap $\leq d$, and thus is in Δ by assumption. If instead $\text{gap}(\sigma) \geq d + 1$, we use the unit-interval assumption: for every i in $S_\sigma \stackrel{\text{def}}{=} \{i \notin \sigma \text{ such that } b_0 < i < b_{d-1}\}$, the d -face $\sigma \cup \{i\}$ is in Δ . So either way the claim is proven. \square

Theorem 60 (Higher-dimensional Chen–Chang–Chang). *Let Δ be a pure d -dimensional simplicial complex.*

- *If Δ is unit-interval and the deletion of $\leq d$ vertices, however chosen, yields a strongly-connected d -complex, then Δ is Hamiltonian.*
- *If Δ is weakly-closed and Hamiltonian, the deletion of ≤ 1 vertices, however chosen, yields a strongly-connected d -complex.*

Proof. For the second claim: Up to a cyclic reshuffling, the vertex we wish to delete is n . The argument of Remark 57 yields a dual path in Δ from each d -face F to H_1 . If F does not contain

n , none of the d -faces in such dual path does, so the path belongs to the dual graph of the deletion of n from Δ .

Now we prove the first claim. By Lemma 59, Δ contains all d -faces of gap $\leq d$. In particular:

- for any odd i such that $1 \leq i \leq n - 2d$, Δ contains the gap- d face O_i formed by i and by the first d consecutive odd integers after i ;
- for any even j such that $2 \leq j \leq n - 2d$, Δ contains the gap- d face E_j formed by j and by the first d consecutive even integers after j ;
- Δ contains the gap- $(d - 1)$ face $F = 1, 2, 4, \dots, 2d$ formed by 1 and by the d smallest even natural numbers;
- Δ contains the gap- $(d - 1)$ face G formed by the largest even integer $\leq n$ and by the d largest odd integers $\leq n$.

Now consider the following sequence \mathfrak{C} of d -faces in Δ : First all O_i 's in increasing order, then G , then all E_j 's in decreasing order, then F . Note that any two O_i 's are adjacent, and the last of them is adjacent to G ; symmetrically, any two E_j 's are adjacent, and F is adjacent to E_2 . We claim that this sequence would form a weakly-Hamiltonian cycle if we relabeled the vertices of Δ first by listing the odd ones increasingly, and then the even ones decreasingly.

Formally, if n is odd, we introduce the new labeling

$$\ell_1 \stackrel{\text{def}}{=} 1, \ell_2 \stackrel{\text{def}}{=} 3, \ell_3 \stackrel{\text{def}}{=} 5, \dots, \ell_{\frac{n+1}{2}} \stackrel{\text{def}}{=} n, \ell_{\frac{n+1}{2}+1} \stackrel{\text{def}}{=} n-1, \ell_{\frac{n+1}{2}+2} \stackrel{\text{def}}{=} n-3, \dots, \ell_{n-1} \stackrel{\text{def}}{=} 4, \ell_n \stackrel{\text{def}}{=} 2.$$

And if instead n is even, we introduce the new labeling

$$\ell_1 \stackrel{\text{def}}{=} 1, \ell_2 \stackrel{\text{def}}{=} 3, \ell_3 \stackrel{\text{def}}{=} 5, \dots, \ell_{\frac{n}{2}} \stackrel{\text{def}}{=} n-1, \ell_{\frac{n}{2}+1} \stackrel{\text{def}}{=} n, \ell_{\frac{n}{2}+2} \stackrel{\text{def}}{=} n-2, \dots, \ell_{n-1} \stackrel{\text{def}}{=} 4, \ell_n \stackrel{\text{def}}{=} 2.$$

Let us set $L_1 \stackrel{\text{def}}{=} \ell_1 \ell_2 \cdots \ell_{d+1}$, $L_2 \stackrel{\text{def}}{=} \ell_2 \ell_3 \cdots \ell_{d+2}$, and so on. Then the sequence \mathfrak{C} described above is equal (whether n is even or odd) to

$$L_1, L_2, \dots, L_{\lfloor \frac{n+1}{2} \rfloor - (d-1)}, L_{\lfloor \frac{n+1}{2} \rfloor + 1}, L_{\lfloor \frac{n+1}{2} \rfloor + 2}, \dots, L_{n-d}, L_{n-(d-1)}.$$

This shows that with the new labeling Δ is *weakly*-Hamiltonian. It remains to show for $d \geq 2$ that our weakly-Hamiltonian cycle can indeed be ‘completed’ to a Hamiltonian cycle, in the sense that the L_i 's that were not mentioned in \mathfrak{C} are anyway contained in Δ . First of all, note that Δ with the original labeling contained all the d -faces of gap $\leq d$, so in particular it contained all d -faces containing 1 and with vertex set contained in $F \cup O_1$. This shows that with the new labeling, $L_{n-(d-2)}, \dots, L_n$ are all in Δ . So it remains to consider the missing L_i 's from the ‘center’ of the sequence \mathfrak{C} . For the “ n odd” case (the case for n even is analogous), we have to see whether Δ contains also the $d - 1$ facets

$$L_{\frac{n+1}{2}-d+2}, L_{\frac{n+1}{2}-d+3}, \dots, L_{\frac{n+1}{2}}.$$

When we translate these d -faces back into the *old* labeling, it is easy to see that the face with the largest gap is the last one, which has gap $d - 1$. So all these faces are in Δ by assumption. \square

Example 61. Let Δ be an unit-interval 3-complex on $n = 9$ vertices that contains all tetrahedra with gap ≤ 3 . With the notation of Theorem 60 the complex Δ contains the sequence \mathfrak{C} below:

$$O_1 = 1357, O_2 = 3579, G = 5789, E_2 = 2468, F = 1246.$$

If we relabel the vertices as in the proof of Theorem 60, the list above becomes

$$L_1, L_2, L_3, L_6, L_7.$$

Thus Δ is weakly-Hamiltonian. To prove that it is Hamiltonian, we need to check that L_4, L_5 and L_8, L_9 are in Δ . Translated into the original labeling, this means checking that 6789, 4689 and 1234, 1235 are in Δ , which is clearly the case because they all have gap ≤ 2 .

Remark 62. For $d = 1$, Theorem 60 boils down to Chen–Chang–Chang’s result that “unit interval graphs are Hamiltonian if and only if they are 2-connected” [CCC97, Theorem 2]. The G_5 of Remark 45 is 2-connected and not Hamiltonian; hence the “unit-interval” assumption in the first claim of Theorem 60 is necessary. As for the second claim, the “weakly-closed” assumption is necessary for $d > 1$, because we saw in Remark 2 that some Hamiltonian d -complexes are not strongly-connected.

We may condense most of the results of this chapter in the following summary:

Theorem 63. *Let Δ be a d -dimensional simplicial complex.*

- (1) *If Δ is closed (or unit-interval) and strongly connected, then Δ is traceable.*
- (2) *If Δ is closed (or unit-interval), and the deletion of d or less vertices, however chosen, yields a strongly connected complex, then Δ is Hamiltonian.*
- (3) *If Δ is under-closed, it contains H_1 . If in addition Δ has a face of minimum i for each $i \in \{2, \dots, n - d\}$, then Δ is traceable.*
- (4) *If Δ is semi-closed, then for every face $F = a_0 \cdots a_d$ of Δ either H_{a_0} or H_{a_d-d} is in Δ .*
- (5) *If Δ is weakly-closed, then Δ contains at least one of the H_i ’s. If in addition Δ contains H_1 , plus a face with minimum i and of gap smaller than d for each i in $\{2, \dots, n - d\}$, then Δ is weakly-traceable.*

Proof. (1) This is given by Proposition 54 and Theorem 56 above.

(2) This is given by Proposition 54 and Theorem 60 above.

(3) By definition of under-closed, if Δ has a face of minimum i , then Δ contains H_i . The fact that Δ has a face of minimum 1 follows from the assumption that Δ is pure.

(4) This is straightforward from the definition of semi-closed.

(5) Let $F = a_0 a_1 \cdots a_d$ be any facet of Δ with $\text{gap}(F) > 0$. Let $g \notin F$ such that $a_0 < g < a_d$. By definition of “weakly-closed”, some face $G = b_0 b_1 \cdots b_d$ of Δ contains g , is adjacent to F , and has either $b_0 \neq a_0$ or $b_d \neq a_d$. Thus $\text{gap} G < \text{gap} F$. Iterating the process, eventually we find in Δ a gap-0 face, which has to be one of

$$H_{a_0}, H_{a_0+1}, \dots, H_{a_d-d}.$$

As for the second claim: By assumption, Δ contains H_1 . Also, Δ contains H_{n-d} , because no other face has minimum $n - d$. Now let $H' = 2a_1 \cdots a_d$ be a face of Δ with minimum 2 and $\text{gap} \leq d - 1$. By the argument above, we know that Δ must contain at least one of

$$H_2, H_3, \dots, H_{a_d-d}.$$

Let us call this face H_{i_2} . By how H' was chosen,

$$2 \leq i_2 \leq a_d - d = \text{gap}(H') + 2 \leq d + 1.$$

But since H_1 contains all vertices from 1 to $d + 1$, in particular it contains i_2 . So H_{i_2} is incident with H_1 . Now let $H'' = a_0 a_1 \cdots a_d$ be a face of Δ with gap smaller than d , and minimum $a_0 = i_2 + 1$. Repeating the argument above, Δ contains one of

$$H_{i_2+1}, H_{i_2+2}, \dots, H_{a_d-d}.$$

Call this facet H_{i_3} ; as above, it must intersect H_{i_2} . And so on. Eventually, we obtain a list $H_1 = H_{i_1}, H_{i_2}, \dots, H_{i_{k-1}}, H_{i_k} = H_{n-d}$ of facets of Δ that makes it weakly-traceable. \square

Remark 64. In the previous theorem, a relabeling was necessary only to prove item (2). For all other items, the *original* labeling was already suitable for the desired conclusion. So for item (1) we proved a slightly stronger statement: “If Δ is strongly-connected, then any labeling that makes Δ unit-interval automatically makes Δ traceable”. Same for items (3), (4), (5).

3 Algebraic motivation

In this section, we review Ene et al's definition of determinantal facet ideals [E&13]. We find out a large class of them that are radical. In fact, we prove the following:

- If a simplicial complex is semi-closed, then its determinantal facet ideal has a square-free Gröbner degeneration (and in particular is radical), and the quotient by such ideal in positive characteristic is F -pure (Theorem 77).
- If the simplicial complex is unit-interval, then the natural generators of its determinantal facet ideal form a Gröbner basis with respect to a diagonal term order (Theorem 82). Moreover, the converse is true if with respect to the same labeling, the simplicial complex is traceable (Theorem 87).

3.1 A foreword on F -pure rings, F -split rings, and Knutson ideals

Let p be a prime number. Let R be a ring of characteristic p . Recall that the *Frobenius map* is the ring homomorphism from R to itself that maps an element $r \in R$ to r^p . We denote by F_*R the R -module defined as follows: $F_*R \stackrel{\text{def}}{=} R$ as additive group, and $r \cdot x \stackrel{\text{def}}{=} r^p x$ for all $r \in R$ and $x \in F_*R$. This allows us to view the Frobenius map as a map of R -modules,

$$\begin{aligned} F : R &\longrightarrow F_*R \\ r &\longmapsto r^p. \end{aligned}$$

The ring R is reduced if and only if F is injective. So the following definitions are natural:

Definition 65. R is F -pure if $F \otimes 1_M : M \rightarrow F_*R \otimes_R M$ is injective for any R -module M .

Definition 66. R is F -split if there exists a homomorphism $\theta : F_*R \rightarrow R$ of R -modules such that $\theta \circ F = 1_R$. Such a θ is called an F -splitting of R .

If a ring is F -split, it is clearly F -pure. The converse does not hold in general. However, the two concepts are equivalent in a number of cases, for example:

Lemma 67. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a Noetherian graded ring of characteristic p having a unique homogeneous ideal \mathfrak{m} that is maximal with respect to inclusion. Furthermore, assume that the Noetherian local ring R_0 is complete. Then the following are equivalent:

- R is F -split.
- R is F -pure.
- $F \otimes 1_E : E \rightarrow F_*R \otimes_R E$ is injective, where E is the injective hull of R/\mathfrak{m} .

Proof. (a) \implies (b) \implies (c) are obvious implications. To see (c) \implies (a): the map

$$F \otimes 1_E : E \longrightarrow F_*R \otimes_R E$$

is injective if and only if the corresponding map

$$\text{Hom}_R(F_*R, \text{Hom}_R(E, E)) \cong \text{Hom}_R(F_*R \otimes_R E, E) \longrightarrow \text{Hom}_R(E, E)$$

is surjective. Hence, by [BH93, Corollary 3.6.7, Proposition 3.6.16, Theorem 3.6.17], the corresponding map $\alpha : \text{Hom}_R(F_*R, R) \rightarrow R$ is surjective. So there exists $\theta \in \text{Hom}_R(F_*R, R)$ such that $\alpha(\theta) = 1$. On the other hand, by construction $\alpha(\theta) = \theta(F(1))$, so $\theta \circ F = 1_R$. \square

Since we want to study homogeneous quotients of a polynomial ring over a field, by Lemma 67 we may as well regard the F -split notion and the F -pure notion as equivalent.

In the following the concept of *Knutson ideal* will be fundamental. The name arises from the work of Knutson [Knu09], later systematically investigated by the second author [Sec20], who extended several properties from $\mathbb{Z}/p\mathbb{Z}$ to any field. The result from [Sec20] that we shall need is the following:

Theorem 68 (Seccia [Sec20]). *Let K be a field. Let $g \in S = K[x_1, \dots, x_n]$ be a polynomial with $\text{in}_{<}(g)$ square-free for some term order on S . Let \mathcal{C}_g be the smallest set of ideals of S containing (g) and such that:*

1. $I \in \mathcal{C}_g \implies I : h \in \mathcal{C}_g$ whenever $h \in S$,
2. $I, J \in \mathcal{C}_g \implies I + J \in \mathcal{C}_g, I \cap J \in \mathcal{C}_g$.

If $I \in \mathcal{C}_g$, then $\text{in}_{<}(I)$, and therefore I , is radical. Furthermore, if $I, J \in \mathcal{C}_g$, then $\text{in}_{<}(I + J) = \text{in}_{<}(I) + \text{in}_{<}(J)$ and $\text{in}_{<}(I \cap J) = \text{in}_{<}(I) \cap \text{in}_{<}(J)$. Finally, if K has positive characteristic, S/I is F -pure whenever $I \in \mathcal{C}_g$.

Example 69. It can be shown that, if $g = x_1 x_2 \cdots x_n$, then \mathcal{C}_g is the set of squarefree monomial ideals.

3.2 Determinantal facet ideals: basic properties

Let d, n be positive integers with $d + 1 \leq n$. Let $S \stackrel{\text{def}}{=} K[x_{ij} : i = 1, \dots, n, j = 0, \dots, d]$ be a polynomial ring in $(d + 1)n$ variables over some field K . Set

$$X = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \\ x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dn} \end{bmatrix}.$$

Given $1 \leq r \leq d$, and integers $0 \leq a_0 < a_1 < \dots < a_r \leq d$ and $1 \leq b_0 < \dots < b_r \leq n$, an $(r + 1)$ -minor of X is any element of the form

$$[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] \stackrel{\text{def}}{=} \det \begin{bmatrix} x_{a_0 b_0} & x_{a_0 b_1} & \cdots & x_{a_0 b_r} \\ x_{a_1 b_0} & x_{a_1 b_1} & \cdots & x_{a_1 b_r} \\ \vdots & \vdots & \cdots & \vdots \\ x_{a_r b_0} & x_{a_r b_1} & \cdots & x_{a_r b_r} \end{bmatrix}.$$

If $r = d$, the row indices are forced to be $a_0 = 0, a_1 = 1, \dots, a_d = d$. For this reason we denote $[01 \dots d | b_0 b_1 \dots b_d]$ simply by $[b_0 b_1 \dots b_d]$. The ideal of S generated by the $r + 1$ -minors of X is denoted by $I_{r+1}(X)$. This ideal defines the variety of $(d + 1)n$ matrices with entries in K and with rank at most r . The set Π of all the minors of X can be partially ordered by the relation

$$[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] \leq [c_0 c_1 \dots c_s | d_0 d_1 \dots d_s] \stackrel{\text{def}}{\iff} r \geq s, \quad a_i \leq c_i \text{ and } b_i \leq d_i \quad \forall i = 0, \dots, s.$$

In particular, for maximal minors the previous definition restricts to

$$[a_0 a_1 \dots a_d] \leq [b_0 b_1 \dots b_d] \iff a_0 \leq b_0, a_1 \leq b_1, \dots, a_d \leq b_d.$$

It is not our intent to review the theory of *Algebras with Straightening Law* here, as the interested reader can learn it directly from the standard source [BV88]. However, we wish to introduce a few concepts for the sake of clarity. The starting observation is that the polynomial ring S is generated by Π as a K -algebra. In fact, a basis of S as K -vector space is given by

$$\{\pi_1 \cdots \pi_m : m \in \mathbb{N}, \pi_i \in \Pi, \pi_1 \leq \pi_2 \leq \dots \leq \pi_m\}.$$

The elements of this K -basis are called *standard monomials*. It may happen that the product of two standard monomials is not a standard monomial. However, such product will be uniquely writable as K -linear combination of standard monomials, which is in some sense compatible with the poset structure on Π . This is what is known as ‘Straightening Law’; compare [BV88, Theorem 4.11]. What we wish to outline is that the ideals of S generated by poset ideals of Π (i.e. subsets $\Omega \subset \Pi$ such that for all $\omega \in \Omega$, $\pi \in \Pi$, $\pi \leq \omega \implies \pi \in \Omega$) are particularly nice.

Example 70. For any $r \leq d$, the ideal $I_{r+1}(X)$ is generated by the poset ideal $\Omega_{\geq r+1}$ of all t -minors of X with $t \geq r+1$. This $\Omega_{\geq r+1}$ has a unique maximal element, $[d-r \dots d|n-r \dots n]$.

Some new notation: if $1 \leq i < j \leq n$, by $X_{[i,j]}$ we mean the matrix

$$X_{[i,j]} = \begin{bmatrix} x_{0i} & x_{0,i+1} & \dots & x_{0j} \\ x_{1i} & x_{1,i+1} & \dots & x_{1j} \\ \vdots & \vdots & \dots & \vdots \\ x_{di} & x_{d,i+1} & \dots & x_{dj} \end{bmatrix},$$

so $I_{r+1}(X_{[i,j]})$ is the ideal of S generated by the $r+1$ -minors of $X_{[i,j]}$, whenever $r \leq \min\{d, j-i\}$.

Eventually, we say that a term order $<$ on S is a *diagonal term order* if, for all $1 \leq r \leq d$ and integers $0 \leq a_0 < a_1 < \dots < a_r \leq d$ and $1 \leq b_0 < \dots < b_r \leq n$, $\text{in}_<([a_0 a_1 \dots a_r | b_0 b_1 \dots b_r]) = x_{a_0 b_0} x_{a_1 b_1} \dots x_{a_r b_r}$. For example, the lexicographic term order on S extending the linear order of the variables given by $x_{ij} > x_{hk}$ if and only if $i < h$ or $i = h$ and $j < k$ is a diagonal term order. We will use the following result from [Stu90]:

Theorem 71 (Sturmfels [Stu90]). *If $<$ is a diagonal term order, $1 \leq i < j \leq n$ and $r \leq \min\{d, j-i\}$, then $\{[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] : 0 \leq a_0 < a_1 < \dots < a_r \leq d \text{ and } i \leq b_0 < \dots < b_r \leq j\}$ is a Gröbner basis of the $I_{r+1}(X_{[i,j]})$.*

So far, by a ‘simplicial complex on n vertices’ we have always implicitly assumed that each vertex $i = 1, \dots, n$ appears in the complex. From now on, we will drop this convention, i.e. henceforth a simplicial complex on a set A is also a simplicial complex on any finite set $B \supset A$.

Definition 72. Let Δ be a d -dimensional simplicial complex on n vertices. Let K be any field. Let $S = K[x_{ij} : i = 1, \dots, n, j = 0, \dots, d]$. The *determinantal facet ideal* of Δ is the ideal

$$J_\Delta := ([a_0 a_1 \dots a_d] : a_0 a_1 \dots a_d \in \Delta) \subset S.$$

When $d = 1$, then Δ is a graph, and J_Δ is the *binomial edge ideal* of Δ . Binomial edge ideals have been intensively studied in the recent literature: Among the many papers on this topic, see for example [H&10], [Oht11], [MM13], [Mat18]. Unlike binomial edge ideals, determinantal facet ideals are not always radical – not even if the complex is weakly-closed:

Example 73. Consider the weakly-closed 2-dimensional simplicial complex on five vertices

$$\Delta = 124, 145, 234, 345.$$

Thus in the polynomial ring with 15 variables $x_{i,j}$, for $i \in \{0, 1, 2\}$ and $j \in \{1, \dots, 5\}$, the ideal J_Δ is generated by the four degree-3 polynomials

$$\begin{aligned} & -x_{0,4}x_{1,2}x_{2,1} + x_{0,2}x_{1,4}x_{2,1} + x_{0,4}x_{1,1}x_{2,2} - x_{0,1}x_{1,4}x_{2,2} - x_{0,2}x_{1,1}x_{2,4} + x_{0,1}x_{1,2}x_{2,4}, \\ & -x_{0,5}x_{1,4}x_{2,1} + x_{0,4}x_{1,5}x_{2,1} + x_{0,5}x_{1,1}x_{2,4} - x_{0,1}x_{1,5}x_{2,4} - x_{0,4}x_{1,1}x_{2,5} + x_{0,1}x_{1,4}x_{2,5}, \\ & -x_{0,4}x_{1,3}x_{2,2} + x_{0,3}x_{1,4}x_{2,2} + x_{0,4}x_{1,2}x_{2,3} - x_{0,2}x_{1,4}x_{2,3} - x_{0,3}x_{1,2}x_{2,4} + x_{0,2}x_{1,3}x_{2,4}, \\ & -x_{0,5}x_{1,4}x_{2,3} + x_{0,4}x_{1,5}x_{2,3} + x_{0,5}x_{1,3}x_{2,4} - x_{0,3}x_{1,5}x_{2,4} - x_{0,4}x_{1,3}x_{2,5} + x_{0,3}x_{1,4}x_{2,5}. \end{aligned}$$

It can be checked using the software Macaulay 2 [GSm2] that J_Δ is not radical.

Determinantal facet ideals are multi-graded. To see this, we endow S with the multi-grading defined by $\deg(x_{ij}) = \mathbf{e}_j \in \mathbb{N}^n$ for all $i = 0, \dots, d$, $j = 1, \dots, n$. Here \mathbf{e}_j is the vector with a one in position j , and zeroes everywhere else. With such grading J_Δ is homogeneous, and S/J_Δ admits a multi-graded minimal free resolution

$$0 \rightarrow \bigoplus_{\mathbf{v} \in \mathbb{N}^n} S(-\mathbf{v})^{\beta_{p,\mathbf{v}}} \rightarrow \dots \rightarrow \bigoplus_{\mathbf{v} \in \mathbb{N}^n} S(-\mathbf{v})^{\beta_{1,\mathbf{v}}} \rightarrow S \rightarrow S/J_\Delta \rightarrow 0,$$

where p is the projective dimension of S/J_Δ . We set $|\mathbf{v}| = v_1 + \dots + v_n$ for each $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{N}^n$; this way the *graded Betti numbers* with respect to the standard grading are

$$\beta_{i,j} = \sum_{\substack{\mathbf{v} \in \mathbb{N}^n \\ |\mathbf{v}|=j}} \beta_{i,\mathbf{v}}.$$

In particular, $\text{reg}(S/J_\Delta) = \max\{|\mathbf{v}| - i : \beta_{i,\mathbf{v}} \neq 0\}$. In the next result, inspired by [MM13, Lemma 2.1] $\text{supp}(\mathbf{v}) = \{i : v_i \neq 0\} \subset [n]$ for each $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{N}^n$. For each subset $W \subset \{1, \dots, n\}$, by Δ_W we denote the subcomplex of Δ induced on W .

Proposition 74. *Let Δ be a d -dimensional simplicial complex on n vertices and $W \subset [n]$. Whenever $\mathbf{v} \in \mathbb{N}^n$ is such that $\text{supp}(\mathbf{v}) \subset W$,*

$$\beta_{i,\mathbf{v}}(S/J_\Delta) = \beta_{i,\mathbf{v}}(S/J_{\Delta_W}) \quad \forall i \in \mathbb{N}.$$

In particular, $\text{reg}(S/J_\Delta) \geq \text{reg}(S/J_{\Delta_W})$.

Proof. Let \mathbb{F} be the multi-graded minimal free resolution of S/J_Δ :

$$\mathbb{F} : 0 \rightarrow \bigoplus_{\mathbf{v} \in \mathbb{N}^n} S(-\mathbf{v})^{\beta_{p,\mathbf{v}}} \rightarrow \dots \rightarrow \bigoplus_{\mathbf{v} \in \mathbb{N}^n} S(-\mathbf{v})^{\beta_{1,\mathbf{v}}} \rightarrow S \rightarrow 0.$$

Consider the complex of multi-graded S -modules

$$\mathbb{F}' : 0 \rightarrow \bigoplus_{\substack{\mathbf{v} \in \mathbb{N}^n \\ \text{supp}(\mathbf{v}) \subset W}} S(-\mathbf{v})^{\beta_{p,\mathbf{v}}} \rightarrow \dots \rightarrow \bigoplus_{\substack{\mathbf{v} \in \mathbb{N}^n \\ \text{supp}(\mathbf{v}) \subset W}} S(-\mathbf{v})^{\beta_{1,\mathbf{v}}} \xrightarrow{\phi} S \rightarrow 0.$$

The cokernel of ϕ is S/J_{Δ_W} , hence all we need to show is that \mathbb{F}' is acyclic. But since the minimal generators of the free S -modules in \mathbb{F}' involve only the variables x_{ij} with $j \in W$, to show that \mathbb{F}' is acyclic is enough to show that $\mathbb{F}'_{\mathbf{u}}$ is acyclic for any $\mathbf{u} \in \mathbb{N}^n$ with $\text{supp}(\mathbf{u}) \subset W$. On the other hand, for any $\mathbf{v} \in \mathbb{N}^n$, $S(-\mathbf{v})_{\mathbf{u}}$ is nonzero if and only if $\mathbf{u} - \mathbf{v} \in \mathbb{N}^n$: in particular $S(-\mathbf{v})_{\mathbf{u}} \neq 0$ implies $\text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u}) \subset W$, hence $\mathbb{F}'_{\mathbf{u}} = \mathbb{F}_{\mathbf{u}}$ whenever $\text{supp}(\mathbf{u}) \subset W$. We conclude since $\mathbb{F}_{\mathbf{u}}$ is acyclic for any $\mathbf{u} \in \mathbb{N}^n$. \square

3.3 Many radical and many F-pure determinantal facet ideals

Let us warm up by studying the algebraic counterpart of the traceability of Δ :

Proposition 75. *Let Δ be a traceable d -dimensional simplicial complex on n vertices. Then $\text{height}(J_\Delta) = n - d$. Furthermore, if J_Δ is radical and unmixed, then it admits a square-free initial ideal. If in addition K has positive characteristic, then S/J_Δ is even F -pure.*

More precisely,

$$\begin{aligned} \boldsymbol{\mu} \stackrel{\text{def}}{=} & [d|1][d-1, d|1, 2] \cdots [1, 2, \dots, d-1, d|1, 2, \dots, d-1, d] \cdot \\ & [1, 2, \dots, d, d+1] \cdots [n-d, n-d+1, \dots, n-1, n] \cdot \\ & [n-d+1, n-d+2, \dots, n-1, n|0, 1, \dots, d-2, d-1] \cdots [n-1, n|0, 1][n|0]. \end{aligned}$$

The reason we defined $\boldsymbol{\mu}$ this way is that if $<$ is a diagonal term order, we have

$$\text{in}_{<}(\boldsymbol{\mu}) = \prod_{i=0}^d \prod_{j=1}^n x_{ij}.$$

Using this $\boldsymbol{\mu}$, we are now ready to prove the first main result of this Chapter.

Theorem 77. *Let Δ be a d -dimensional semi-closed simplicial complex on n vertices. Then J_Δ is a radical ideal. Moreover:*

- (1) *For any diagonal term order (compatible with the labeling which makes Δ semi-closed), $\text{in}(J_\Delta)$ is a squarefree term ideal.*
- (2) *If the field K has positive characteristic, S/J_Δ is F -pure.*

Proof. We will prove that if Δ is semi-closed with respect to the given labeling then $J(\Delta) \in \mathcal{C}_\boldsymbol{\mu}$, whence both claims follow by Theorem 68. Let $1 \leq a_0 < a_1 < \dots < a_d \leq n$. Using the notation of Lemma 76, since Δ is semi-closed, either $\Gamma_{\mathbf{a}}$ or $\Gamma^{\mathbf{a}}$ is contained in Δ whenever $a_0 a_1 \cdots a_d \in \Delta$. For any $a_0 a_1 \cdots a_d \in \Delta$, set $\Delta_{\mathbf{a}} = \Gamma_{\mathbf{a}}$ if $\Gamma_{\mathbf{a}} \subset \Delta$, and $\Delta_{\mathbf{a}} = \Gamma^{\mathbf{a}}$ otherwise. Then

$$\Delta = \bigcup_{a_0 a_1 \cdots a_d \in \Delta} \Delta_{\mathbf{a}}.$$

In particular,

$$J(\Delta) = \sum_{a_0 a_1 \cdots a_d \in \Delta} J(\Delta_{\mathbf{a}}).$$

Since $\mathcal{C}_\boldsymbol{\mu}$ is closed under sums, in order to show that $J(\Delta) \in \mathcal{C}_\boldsymbol{\mu}$ we only need to check that each $J(\Delta_{\mathbf{a}}) \in \mathcal{C}_\boldsymbol{\mu}$. To verify this, we use a result in [Sec21]: The ideal $I_{r+1}(X_{[ij]}) \in \mathcal{C}_\boldsymbol{\mu}$ whenever $1 \leq i < j \leq n$ and $0 \leq r \leq \min\{d, j-i\}$. Since $\mathcal{C}_\boldsymbol{\mu}$ is closed under intersections, Lemma 76 guarantees that $J(\Delta_{\mathbf{a}}) \in \mathcal{C}_\boldsymbol{\mu}$, as desired. \square

Remark 78. The assumption “semi-closed” is best possible: if we replace it with “weakly-closed”, the theorem no longer holds, cf. Example 73. That said, the converse of Theorem 77 is false. To see this, consider the non-weakly-closed complex $U_3^2 = 124, 345, 467$ of Figure 3. If $g = [124][345][467]$ then for a diagonal term order $\text{in}(g) = x_{01}x_{12}x_{24}x_{03}x_{14}x_{25}x_{04}x_{16}x_{27}$, which is squarefree. Obviously $[124], [345], [467] \in \mathcal{C}_g$, hence $J_\Delta \in \mathcal{C}_g$. So $\text{in}(J_\Delta)$ is squarefree, and, in the positive characteristic case, S/J_Δ is F -pure by Theorem 68. On the other hand, when $d = 1$ Theorem 77 is true for all weakly closed graphs, via the main result of Matsuda [Mat18]. This shows that the techniques used in [Mat18] do not generalize to higher dimensions.

Remark 79. Suppose that K has positive characteristic. Theorem 77 implies that, whenever Ω is a poset ideal of Π consisting only of maximal minors, then the corresponding ASL is F -pure. On the other hand, some ASLs are not F -pure, as explained in [KV21, Remark 5.2]. We do not know whether all the ASLs on a poset ideal of Π are F -pure.

Remark 80. In positive characteristic, having a square-free initial ideal or an F -pure quotient are unrelated properties. Many ideals, like $I = (x^2 + xy + y^2) \subset S \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}[x, y]$, for p prime, have the property that S/I is F -pure even if $\text{in}(I)$ is not square-free for any term order. On the other hand, the binomial edge ideal of a 5-cycle is not F -pure in characteristic 2 [Mat18, Example 2.7], even if it admits a squarefree initial ideal. See [KV21] for a discussion on the relationship between the two properties of being F -pure and having a squarefree initial ideal.

Theorem 77 allows us to characterize the determinantal facet ideals having a linear resolution: It turns out that there is only one. This extends to all dimensions the result for graphs by Saeedi-Madani and Kiani [SK12].

Corollary 81. *Let Δ be a pure d -dimensional simplicial complex on n vertices.*

$$J_\Delta \text{ has a linear resolution} \iff \Delta = \Sigma_n^d.$$

Proof. “ \Leftarrow ”: If Δ is the d -skeleton of the $(n - 1)$ -simplex, J_Δ is the ideal of maximal minors of the matrix X . This ideal is resolved by the Eagon-Northcott complex [EN67], which is linear.

“ \Rightarrow ”: By contradiction, suppose there is a subset $W \subset [n]$ of cardinality $d + 2$ such that Δ_W is not the d -skeleton of the $(d + 1)$ -simplex on W . We can re-label the vertices so that $W = \{1, 2, \dots, d + 2\}$ and

$$\Delta_W = 12 \dots (d + 1), 12 \dots d(d + 2), \dots, 1 \dots i(i + 2)(i + 3) \dots (d + 2)$$

where $2 \leq i \leq d$. With respect to such a labeling Δ_W is semi-closed. So by Theorem 77, $\text{in}(J_{\Delta_W})$ is a squarefree monomial ideal for any diagonal term order. Hence, by the work of Conca–Varbaro [CV20], $\text{reg}(S/J_{\Delta_W}) = \text{reg}(S/\text{in}(J_{\Delta_W}))$. But by Lemma 76

$$J_{\Delta_W} = I_i(X_{[1,i]}) \cap I_{d+1}(X_{[1,d+2]}),$$

so by Theorem 68 $\text{in}(J_{\Delta_W}) = \text{in}(I_i(X_{[1,i]})) \cap \text{in}(I_{d+1}(X_{[1,d+2]}))$. Via Theorem 71, it is easy to check that the monomial $(x_{d-i+1,1}x_{d-i+2,2} \cdots x_{d,i})(x_{0,2}x_{1,3} \cdots x_{d,d+2})$ is a minimal generator of $\text{in}(I_i(X_{[1,i]})) \cap \text{in}(I_{d+1}(X))$. Hence $\text{in}(J_{\Delta_W})$ has a minimal generator of degree $i + d + 1$. In particular,

$$\text{reg}(S/J_\Delta) \geq \text{reg}(S/J_{\Delta_W}) = \text{reg}(\text{in}(S/J_{\Delta_W})) \geq i + d > d.$$

So by Proposition 74, $\text{reg}(S/J_\Delta) \geq \text{reg}(S/J_{\Delta_W}) > d$. So J_Δ cannot have a linear resolution. \square

3.4 Determinantal facet ideals defined by a Gröbner basis

If Δ is a closed simplicial complex, it is easy to see that the minors generating J_Δ form a Gröbner basis with respect to a diagonal monomial order, corresponding to the labeling that makes Δ closed: See [E&13]. In [E&13] it has been incorrectly claimed that the converse of the above statement holds true. The following result, which is a consequence of [Sec21, Corollary 2.4], shows that there are many other complexes Δ for which the minors generating J_Δ form a Gröbner basis:

Theorem 82. *Let Δ be a d -dimensional simplicial complex, with a labeling that makes it unit-interval. The set $\{[a_0 \dots a_d] : a_0 \dots a_d\}$ is a Gröbner basis of J_Δ with respect to any diagonal term order. If in addition the field K has positive characteristic, then S/J_Δ is F -pure.*

Proof. By definition, Δ is the union of d -skeleta of simplices on consecutive vertices. We can choose these d -skeleta to be maximal with respect to inclusion. This yields a decomposition

$$\Delta = \Sigma_{[i_1, j_1]}^d \cup \Sigma_{[i_2, j_2]}^d \cup \dots \cup \Sigma_{[i_l, j_l]}^d,$$

where $\Sigma_{[i_k, j_k]}^d$ denotes the d -skeleton of the simplex on vertices $i_k, i_k + 1, i_k + 2, \dots, j_k$. Therefore

$$J_\Delta = I_{d+1}(X_{[i_1, j_1]}) + I_{d+1}(X_{[i_2, j_2]}) + \dots + I_{d+1}(X_{[i_l, j_l]}).$$

So by [Sec21, Corollary 2.4]

$$\text{in}_<(J_\Delta) = \text{in}_<(I_{d+1}(X_{[i_1, j_1]})) + \text{in}_<(I_{d+1}(X_{[i_2, j_2]})) + \dots + \text{in}_<(I_{d+1}(X_{[i_l, j_l]})).$$

By Theorem 71, $\{[a_0, \dots, a_d] \mid a_0 \dots a_d \in \Delta\}$ is a Gröbner basis for J_Δ . Finally, the F -purity claim in the case of positive characteristic follows again from [Sec21, Corollary 2.4]. \square

Remark 83. That the set $\{[a_0 \dots a_d] : a_0 \dots a_d \in \Delta\}$ is a Gröbner basis when Δ is unit-interval has been independently proved, using a completely different method, in Almousa–Vandebogert [AV21, Theorem 2.16]. They also obtained the analogous result for r -determinantal facet ideals (a more general concept than determinantal facet ideals) of unit-interval simplicial complexes. We were not aware of the paper [AV21] of Almousa and Vandebogert before posting the first version of the present work on the arXiv. (We coordinated efforts to adopt the same name “unit-interval complexes” in the two papers.) For the sake of completeness, we point out that [Sec21, Corollary 2.4] implies that also r -determinantal facet ideals of unit-interval simplicial complexes define F -pure quotients in positive characteristic. We do not know, however, whether the (r -)determinantal facet ideals of “lcm-closed” complexes, as defined in [AV21], or whether those of “closed complexes”, as defined here, are all F -pure.

Remark 84. The converse of Theorem 82 is false: as explained above, any closed but not unit-interval complex is a counterexample. For a more interesting example, consider

$$W = 123, 124, 134, 234, 235, 245, 345, 568, 789, 81011$$

corresponding to a one-point union of the B^2 and the U_3^2 of Figure 3. This complex W is not unit-interval, not closed, and not even weakly-closed [Pav21]. However, one can verify with Macaulay2 [GSm2] or via [AV21, Theorem 2.15] that $\{[a_0, a_1, a_2] : a_0 \dots a_d \in \Delta\}$ form a Gröbner basis of J_W for any diagonal term order.

Remark 85. Two of the results of [E&13] are incorrect because of the following counterexamples. As we already mentioned, the complex B_d of Lemma 42 (cf. Figure 3) is not closed, but the set of all the minors $[abc]$, where abc ranges over all facets of B^d , is a Gröbner basis of J_{B^d} for any diagonal term order by Theorem 82. Thus one direction of [E&13, Theorem 1.1] is incorrect for all $d > 1$. Moreover, the graph $G_0 = 12, 13, 23, 24, 34$ is closed, but one can verify that S/J_{G_0} is not Cohen-Macaulay. Thus [E&13, Corollary 1.3] is incorrect already for $d = 1$.

The final part of our work is dedicated to the delicate quest for some partial converse for Theorem 82. To increase the chances of success, we restrict ourselves to *traceable* complexes. The traceable assumption is rather natural in this case, as we have anyway seen in Theorem 56 that all strongly-connected unit-interval complexes are traceable. We start off with a Lemma:

Lemma 86. *Let Δ be a simplicial complex such that $\mathcal{GB} \stackrel{\text{def}}{=} \{[a_0, \dots, a_d] \mid a_0 \dots a_d \in \Delta\}$ is a Gröbner basis of J_Δ for some diagonal term order. Let $F = a_0 \dots a_d$ and $G = b_0 \dots b_d$ be two facets of Δ . If for some integer $l \in \{0, \dots, d-1\}$*

- (i) $a_i = b_i$ for all $i \in \{0, \dots, l\}$,
- (ii) $a_{l+1} > a_l + 1$,
- (iii) $b_{l+k} = b_l + k$ for all $k \geq 1$,

then the facet $a_0 \dots a_{l-1}(a_l + 1) a_{l+1} \dots a_d$ is also in Δ . Symmetrically, if for some $l \in \{1, \dots, d\}$

- (iv) $a_i = b_i$ for all $i \in \{l, \dots, d\}$,
- (v) $a_{l-1} < a_l - 1$,
- (vi) $b_{l-k} = b_l - k$ for all $k \in \{1, \dots, l\}$,

then the facet $a_0 \dots a_{l-1}(a_l - 1) a_{l+1} \dots a_d$ is also in Δ .

Proof. It is harmless to assume that the term order $<$ is the lexicographic term order defined before, cf. Theorem 71. Let F and G be two facets of Δ satisfying (i), (ii) and (iii). Let us compute the initial term of the polynomial

$$f \stackrel{\text{def}}{=} [l+1 \dots d \mid a_{l+1} \dots a_d][b_0 \dots b_d] - [l+1 \dots d \mid b_{l+1} \dots b_d][a_0 \dots a_d].$$

If we set

$$\begin{aligned} p &\stackrel{\text{def}}{=} [a_0 \dots a_d], & p' &\stackrel{\text{def}}{=} [l+1 \dots d \mid a_{l+1} \dots a_d] \\ q &\stackrel{\text{def}}{=} [b_0 \dots b_d], & q' &\stackrel{\text{def}}{=} [l+1 \dots d \mid b_{l+1} \dots b_d] \end{aligned}$$

then $f = p'q - pq'$, and by Laplace expansion we have

$$\begin{aligned} p'q &= \underbrace{(x_{0b_0} \cdots x_{l-1b_{l-1}} x_{lb_l} p'q')}_h + g_1, & \mu_1 &< \alpha \ \forall \mu_1 \in \text{supp}(g_1), \ \forall \alpha \in \text{supp}(h), \\ pq' &= \underbrace{(x_{0a_0} \cdots x_{l-1a_{l-1}} x_{la_l} p'q')}_h + g_2, & \mu_2 &< \alpha \ \forall \mu_2 \in \text{supp}(g_2), \ \forall \alpha \in \text{supp}(h). \end{aligned}$$

Furthermore

$$\begin{aligned} \text{in}_{<}(g_1) &= (x_{l+1a_{l+1}} \cdots x_{da_d})(x_{0b_0} \cdots x_{l-1b_{l-1}} x_{lb_{l+1}} x_{l+1b_l} x_{l+2b_{l+2}} \cdots x_{db_d}), \\ \text{in}_{<}(g_2) &= (x_{l+1b_{l+1}} \cdots x_{db_d})(x_{0a_0} \cdots x_{l-1a_{l-1}} x_{la_{l+1}} x_{l+1a_l} x_{l+2a_{l+2}} \cdots x_{da_d}). \end{aligned}$$

Since $\text{in}_{<}(g_2)$ is smaller than $\text{in}_{<}(g_1)$, we conclude that

$$\text{in}_{<}(f) = \text{in}_{<}(g_1 - g_2) = (x_{l+1a_{l+1}} \cdots x_{da_d})(x_{0b_0} \cdots x_{l-1b_{l-1}} x_{lb_{l+1}} x_{l+1b_l} x_{l+2b_{l+2}} \cdots x_{db_d}).$$

In addition $f \in J_\Delta$ because $F, G \in \Delta$. Thus, there must be a minor $g = [c_0 \dots c_d]$ in \mathcal{GB} such that $\text{in}_{<}(g)$ divides $\text{in}_{<}(f)$. Note that for c_0, \dots, c_l we only have one option, namely,

$$\begin{cases} c_0 &= b_0 = a_0 \\ &\vdots \\ c_{l-1} &= b_{l-1} = a_{l-1} \\ c_l &= b_{l+1} = b_l + 1 = a_l + 1. \end{cases}$$

For c_{l+1} we have a priori two possibilities: either $c_{l+1} = b_l$ or $c_{l+1} = a_{l+1}$. But $b_l < b_{l+1} = c_l$, so it must be $c_{l+1} = a_{l+1}$. Similarly, for c_{l+2} we have a priori two options: Either $c_{l+2} = b_{l+2}$, or $c_{l+2} = a_{l+2}$. But by the assumptions, we have that $b_{l+2} \leq a_{l+1} = c_{l+1}$, so since $c_{l+2} > c_{l+1}$ it must be $c_{l+2} = a_{l+2}$. In general, for any $k \geq 2$ we have $b_{l+k} \leq a_{l+k-1} = c_{l+k-1}$. Since $c_i > c_{i-1}$, arguing recursively we obtain that the only possible option is $c_{l+k} = a_{l+k}$ for all $k \geq 2$. Hence we have proved that

$$g = [c_0, \dots, c_d] = [a_0 \dots a_{l-1}(a_l + 1)a_{l+1} \dots a_d].$$

Since g is an element of \mathcal{GB} , we conclude that $a_0 \dots a_{l-1}(a_l + 1)a_{l+1} \dots a_d \in \Delta$.

The proof of the second part of the lemma is symmetric; namely, one considers the polynomial

$$f' \stackrel{\text{def}}{=} [0 \dots l-1 \mid a_0 \dots a_{l-1}][b_0 \dots b_d] - [0 \dots l-1 \mid b_0 \dots b_{l-1}][a_0 \dots a_d] \in J_\Delta$$

whose leading term is

$$\text{in}_{<}(f') = (x_{0a_0} \cdots x_{l-1a_{l-1}})(x_{0b_0} \cdots x_{l-2b_{l-2}}x_{l-1b_l}x_{lb_{l-1}}x_{l+1b_{l+1}} \cdots x_{db_d}),$$

and one proceeds analogously to the argument above. \square

Theorem 87. *Let Δ be a d -dimensional simplicial complex. If with respect to the same labeling Δ is traceable and the set $\{[a_0 \dots a_d] : a_0 \dots a_d\}$ is a Gröbner basis of J_Δ with respect to some diagonal term order, then such labeling makes Δ unit-interval.*

Proof. Let $F = a_0 \dots a_d$ be a facet of Δ with $\text{gap}(F) = k$. We proceed by induction on k . For $k = 0$ there is nothing to prove, so we assume $k > 0$. Let g_1, \dots, g_k be the vertices not in F , and such that $a_0 < g_1 < \dots < g_k < a_d$. We want to show that Δ contains the d -skeleton of $\{a_0, \dots, a_d, g_1, \dots, g_k\}$. The strategy is to first show that $\Sigma_{[a_0+1, a_d]}^d, \Sigma_{[a_0, a_d-1]}^d \subseteq \Delta$ by inductive assumption, and then to prove that Δ contains also the facets of the form $a_0c_1 \dots c_{d-1}a_d$. So let us proceed. Let l be the greatest integer such that $a_l < g_1$, so that $g_1 = a_l + 1$. Consider the two facets F and H_{a_0} of Δ . They satisfy the assumptions of Lemma 86, so

$$F' = a_0 \dots a_{l-1} g_1 a_{l+1} \dots a_d \in \Delta.$$

If $l = 0$, then $\text{gap}(F') = k - 1$, so by the inductive assumption $\Sigma_{[g_1, a_d]}^d = \Sigma_{[a_0+1, a_d]}^d \subseteq \Delta$. Otherwise, since $\text{gap}(F') = k$, we cannot apply the inductive assumption yet. However, we have “shifted” the first gap to the left and now the first missing vertex is $a_l = a_{l-1} + 1$. We can apply again Lemma 86 to the facets F' and H_{a_0} and we get

$$F'' = a_0 \dots a_{l-2} a_l g_1 a_{l+1} \dots a_d \in \Delta.$$

If $l = 1$, then $\text{gap}(F'') = k - 1$, so by the inductive assumption $\Sigma_{[a_1, a_d]}^d = \Sigma_{[a_0+1, a_d]}^d \subseteq \Delta$. Otherwise, once again $\text{gap}(F'') = k$ and the first missing vertex $a_{l-1} = a_{l-2} + 1$ has been shifted by one to the left. Iterating this procedure, we eventually get that

$$(a_0 + 1) \dots a_l g_1 a_{l+1} \dots a_d \in \Delta.$$

This face has gap equal to $k - 1$ and we can finally apply induction: We get $\Sigma_{[a_0+1, a_d]}^d \subseteq \Delta$.

To prove that $\Sigma_{[a_0, a_d-1]}^d \subseteq \Delta$ we use a similar argument. Let l be the smaller integer such that $g_k < a_l$, so that $g_k = a_l - 1$, and consider the two facets of Δ

$$F = a_0 \dots a_{l-1} a_l \dots a_d$$

$$H^{a_d} \stackrel{\text{def}}{=} H_{a_d-d} = (a_d - d)(a_d - d + 1) \dots g_k a_l \dots a_d.$$

Iteratively applying the second part of Lemma 86, we can shift the last missing vertex to the right until we reach the facet

$$a_0 \dots a_{l-1} g_k a_l \dots a_d - 1 \in \Delta,$$

which has gap $k - 1$. So by induction $\Sigma_{[a_0, a_d-1]}^d \subseteq \Delta$.

It remains to prove that all the facets of the form $G = a_0c_1 \dots c_{d-1}a_d$ are in Δ . To do so, we start from $F = a_0a_1 \dots a_d$ and we replace one by one each a_i with the corresponding c_i . In detail: For $i = 1$, we have three possibilities:

- $c_1 = a_1$, or
- $a_0 < c_1 < a_1$, or
- $c_1 > a_1$.

If $c_1 = a_1$ there is nothing to do. If $a_0 < c_1 < a_1$, consider the two facets

$$\begin{aligned} F &= a_0 \ a_1 \dots a_d \\ \tilde{F} &= (a_1 - 1) \ a_1 \dots a_d. \end{aligned}$$

Since $a_0 < c_1 < a_1$, we have that $a_1 - 1 > a_0$. Hence $\tilde{F} \in \Sigma_{[a_0+1, a_d]}^d \subseteq \Delta$. So by Lemma 86 $a_0(a_1 - 1)a_2 \dots a_d \in \Delta$. If $c_1 = a_1 - 1$ we stop, otherwise we repeat the same argument. At each iteration of Lemma 86, the second vertex in the facet decreases by one unit; eventually, we obtain that $a_0 c_1 a_2 \dots a_d \in \Delta$.

As for the third possibility ($c_1 > a_1$), we claim that we can simply dismiss it without loss of generality. In fact, for every $i \in \{1, \dots, d\}$, we can always “flatten all the vertices after a_{i-1} to the right”: that is, we can always replace F with another face in Δ of the form

$$F_i = a_0 a_1 \dots a_{i-1} (a_d - d - i) \dots (a_d - 1) a_d.$$

To see it, let $0 \leq l \leq d-1$ be the largest index for which $a_l + 1 < a_{l+1}$ (such an l must exist because $\text{gap}(F) > 0$). Applying Lemma 86 to the facets F and $\tilde{F} = a_0 \dots a_l (a_l + 1) (a_l + 2) \dots a_l + (d - l)$ in $\Sigma_{[a_0, a_d-1]}^d \subseteq \Delta$, we get that the facet $a_0 \dots a_{l-1} (a_l + 1) a_{l+1} \dots a_d$ is in Δ . Proceeding this way we end up with the face

$$F_l = a_0 \dots a_{l-1} (a_{l+1} - 1) a_{l+1} \dots a_d \in \Delta.$$

Replacing F with F_l , and arguing the same way, we infer that $F_i \in \Delta$ for all $i = 0, \dots, d-1$. In particular, for $i = 1$, we could replace F with a face with same minimum and maximum

$$F_1 = a_0 (a_d - d + 1) (a_d - d + 2) \dots a_d \in \Delta.$$

Note that $c_1 \leq a_d - d + 1$. So our claim is proven: Up to replacing F with F_1 , we can assume that $c_1 \leq a_1$.

So the case $i = 1$ is settled. Consider now $i = 2$. If $c_2 = a_2$, there is nothing to do. Otherwise, flattening the vertices after c_1 of $a_0 c_1 a_2 \dots a_d$ to the right, we may assume that $c_2 < a_2$. Consider the two facets

$$\begin{aligned} F &= a_0 \quad c_1 \quad a_2 \dots a_d \in \Delta \\ \tilde{F} &= (a_2 - 2) (a_2 - 1) a_2 \dots a_d \in \Delta. \end{aligned}$$

Since $c_2 < a_2$, we have that $c_1 < a_2 - 1$, so applying Lemma 86 we obtain that

$$a_0 c_1 (a_2 - 1) a_3 \dots a_d \in \Delta.$$

If $c_2 = a_2 - 1$ we stop, otherwise we repeat the same argument. At every iteration of Lemma 86, the third vertex in the facet decreases by one unit; eventually, we obtain that $a_0 c_1 c_2 a_3 \dots a_d \in \Delta$.

Iterating this procedure for all i 's, we conclude that

$$G = a_0 c_1 c_2 \dots c_{d-1} a_d \in \Delta. \quad \square$$

Remark 88. Very recently Almousa and Vandebogert [AV21] introduced a technical property of simplicial complexes, called “lcm-closed”, that simultaneously generalizes the two properties of being “closed” and being “unit-interval”. They asked [AV21, Question 2.19] whether such property for simplicial complexes would characterize the fact that the minors of the determinantal facet ideal form a Gröbner basis with respect to any diagonal term order. With a little ingenuity, one can see that for traceable complexes, “lcm-closed” is simply equivalent to “unit-closed”. Thus Theorem 87 answers Almousa–Vandebogert’s question positively, for complexes that with respect to the same labeling are traceable.

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