HERZOG IDEALS AND F-SINGULARITIES

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Abstract. In this paper we study the connection between Herzog ideals (i.e., ideals with a squarefree Gröbner degeneration) and F-singularities. More precisely, we show that, in positive characteristic, homogeneous Herzog ideals define F-anti-nilpotent rings, and we inquire, in characteristic 0, on a surprising relationship between being Herzog ideals after a change of coordinates and defining rings of dense *open* F-pure type.

1. Introduction

Let $S = k[X_0, \dots, X_n]$ be a polynomial ring over a field k. Following [14], an ideal $I \subseteq S$ is called a *Herzog ideal* if there exists a monomial order $<$ (i.e., a total order on the monomials of S such that $1 < v$ and $u < v \implies uw < vw$ where u, v, w are arbitrary monomials of S with $v \neq 1$) such that $\mathrm{in}_<(I)$ is squarefree. The class of Herzog ideals is largely populated: It includes ideals defining Algebras with Straightening Law, Knutson ideals, Cartwright-Sturmfels ideals, and many more. The name comes from Herzog's conjecture, resolved in [3], saying that the connection between I and $\mathrm{in}_<(I)$ is much tighter than usual if $\mathrm{in}_<(I)$ is squarefree.

The first purpose of this paper is to study the connection between Herzog ideals and F-singularities; this has already been investigated in [15], where it was proved, for example, that in positive characteristic Herzog ideals define F-full and F-injective rings, see also [11] for the latter result. The proofs of these results are based on ideas in [20], where the stronger notion of F-anti-nilpotent has been introduced. An obstacle to proving F-anti-nilpotency of quotients by Herzog ideals is that it is not known whether this property localizes, see Remark 2.6. In Section 2 we will prove that *homogeneous* (not necessarily with respect to a standard grading) Herzog ideals $I \subseteq S$ define F-anti-nilpotent rings.

Theorem 1.1 (Theorem 2.12). *Let R be an $\mathbb{N}$-graded algebra over an F-finite field k of prime characteristic p with homogeneous maximal ideal $\mathfrak{m}$. Write $R = S/I$ where S is an $\mathbb{N}$-graded polynomial ring over k. If I is a Herzog ideal, then $R_\mathfrak{m}$ is F-anti-nilpotent.*

We next investigate the relationship between Herzog ideals and F-purity. Note that, F-purity implies F-anti-nilpotency by the main result of [20] (see also Theorem 4.1 for a stronger statement). In general, Herzog ideals do not necessarily define F-pure quotients (see Remark 3.2). However, our experiments may suggest some surprising connection (in characteristic 0) between Herzog ideals after a change of coordinates and ideals defining F-pure quotients for *all* primes $p \gg 0$. More precisely, we ask and study the following.

Question 1.2. *Let $I \subseteq \mathbb{Q}[X_0, \dots, X_n]$ be a homogeneous ideal. Consider the following two conditions:*

(1) $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ is F-pure for all primes $p \gg 0$.

(2) Possibly after a change of variables, $I_\mathbb{C} \subseteq \mathbb{C}[X_0, \dots, X_n]$ is a Herzog ideal.

Under what assumptions are the two conditions above equivalent?

Of course, one cannot expect the implication (2) $\implies$ (1) without Gorenstein-type assumptions (see Remark 3.2). On the other hand, we were not aware of any counterexample to the implication (1) $\implies$ (2). Our results in Section 3 address Question 1.2 in the following cases:

Theorem 1.3 (Proposition 3.1). *Let* $I \subseteq \mathbb{Q}[X_0, \dots, X_n]$ *be a homogeneous ideal such that* $I_{\mathbb{C}}$ *defines a projective (connected) nonsingular curve* $C \subseteq \mathbb{P}^n$ *containing at least one* $\mathbb{Q}$*-rational point* $P \in C$*. Then* (1) *and* (2) *in Question 1.2 are equivalent.*

Theorem 1.4 (Theorem 3.3). *Let* $f \in \mathbb{Q}[X_0, \dots, X_n]$ *be a homogeneous polynomial of degree* ≤ 3*. Set* $I = (f)$ *and* $H = \operatorname{Proj}(\mathbb{C}[X_0, \dots, X_n]/I_{\mathbb{C}})$*. Then* (1) *and* (2) *in Question 1.2 are equivalent assuming either* H *is klt or* H *is a curve.*

Finally, in Section 4, we prove some miscellaneous results on the annihilators of F-stable submodules of local cohomology modules. We will prove an extension of the main theorem of [20], and we will prove the following generalization of some results in [7].

Theorem 1.5 (Proposition 4.3). *Let* $(R, \mathfrak{m})$ *be an* F*-finite local ring of prime characteristic* p*, and* $i \in \mathbb{Z}$*. If the annihilator* J *of an* F*-stable subquotient of* $H^i_{\mathfrak{m}}(R)$ *is radical (e.g., if the Frobenius action on that subquotient is injective), then* J *is a uniformly compatible ideal. In particular, if* R *is* F*-anti-nilpotent, then the annihilator of any* F*-stable subquotient of* $H^i_{\mathfrak{m}}(R)$ *is uniformly compatible.*

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2. Positive characteristic preliminaries

Throughout this paper, unless otherwise stated, all rings will be commutative, Noetherian, with multiplicative identity 1. Let R be a ring of prime characteristic p. For $e \in \mathbb{N}$, let $F^e : R \to R$ be the e-th iterates of the Frobenius endomorphism on R, that is, the p^e-th power map. For an R-module M, we denote by $F^e_* M$ the corresponding R-module given by restriction of scalars via F^e.

Definition 2.1. *With notations as above, we say* R *is*

- F*-finite if* $F^e_* R$ *is a finitely generated* R*-module for some (equivalently, all)* $e \in \mathbb{N}$*.*
- F*-pure if the Frobenius map is pure, i.e., if for any* R*-module* M *the induced map* $M \to F_* R \otimes_R M$ *is injective.*
- F*-split if the Frobenius map splits, i.e., if there exists* $\phi \in \operatorname{Hom}_R(F_* R, R)$ *such that* $\phi(r) = r$ *for all* $r \in R$*.*

It is easy to see that F-split always implies F-pure. Moreover, the two notions are equivalent for F-finite rings and for complete local rings, but they may differ in general. If R is a finitely generated algebra over a field k or more generally a complete local ring with coefficient field k, we point out that R is F-finite if and only if $[k : k^p] < \infty$. We refer the readers to [21, Chapters 1 and 2] for these facts.

2.1. **Modules with a Frobenius action.** We use the same notations as above.

Definition 2.2. *A Frobenius action on an* R*-module* M *is an additive map* $F : M \to M$ *such that* $F(rx) = r^pF(x)$ *for all* $r \in R$ *and all* $x \in M$*. An* R*-submodule* $N \subseteq M$ *is said to be* F*-stable if* $F(N) \subseteq N$*.*

Note that giving a Frobenius action on M is the same as giving an R-linear map $M \to F_*M$. For $e \in \mathbb{N}$, let $\mathcal{F}^e_R(-)$ be the Frobenius functor of Peskine-Szpiro on the category of R-modules, that is, the functor defined as the base change to F^e_*R followed by the identification of F^e_*R with R. It is easy to see that giving a Frobenius action on M is equivalent to giving an R-linear map $\mathcal{F}_R(M) \to M$: before identifying F_*R with R, this map is given by $F_*r \otimes x \mapsto rF(x)$.

Let $R[F]$ be the Frobenius skew polynomial ring, i.e., the non-commutative ring generated over R by the symbols $1, F, F^2 \ldots$ by requiring that $Fr = r^pF$ for all $r \in R$. Then having a Frobenius action is the same as being a left $R[F]$-module; moreover an R-submodule is F-stable if and only if it is a left $R[F]$-submodule.[1]

Definition 2.3. *If* R *is* $\mathbb{N}$*-graded, then a graded* $R[F]$*-module* M *is a graded* R*-module* M *such that* $F(M_d) \subseteq M_{pd}$ *for all* $d \in \mathbb{Z}$*.*

Note that if M is a graded $R[F]$-module then the associated R-linear map $\mathcal{F}_R(M) \to M$ is degree preserving.

Definition 2.4. *Let* M *be an* $R[F]$*-module. We say that* M *is*

- *full if the map* $\mathcal{F}^e_R(M) \to M$ *is surjective for some (equivalently, all)* $e \in \mathbb{N}$*.*
- *anti-nilpotent if, for any* $R[F]$*-submodule* $N \subseteq M$*, the induced Frobenius action* $F : M/N \to M/N$ *is injective.*

We remark that M is anti-nilpotent if and only if every $R[F]$-submodule of M is full, see [22, Lemma 2.2]. In particular, if M is anti-nilpotent then it is full.

2.2. **Local cohomology modules and F-singularities.** Given an ideal $I = (f_1, \ldots, f_t) \subseteq R$ and an R-module M, we recall that the i-th local cohomology modules $H^i_I(M)$ can be defined as the i-th cohomology of the Čech complex

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^t M_{f_i} \longrightarrow \bigoplus_{1\leq i<j\leq t} M_{f_if_j} \longrightarrow \ldots\ldots \longrightarrow M_{f_1\cdots f_t} \longrightarrow 0.$$

Moreover, the Frobenius map on R induces maps $F^e : H^i_I(R) \to H^i_{I^{[p^e]}}(R) \cong H^i_I(R)$ for all $i \in \mathbb{Z}$ and all $e \in \mathbb{N}$, where $I^{[p^e]} = F^e(I)R = (f_1^{p^e}, \ldots, f_t^{p^e})$ denotes the e-th Frobenius power of I. Equivalently, they can be seen as R-linear maps $H^i_I(R) \to H^i_I(F^e_*R) \cong F^e_*(H^i_I(R))$.

Definition 2.5. *Let* $(R, \mathfrak{m})$ *be a local ring of prime characteristic* p*. We say that* R *is*

- F*-injective if the induced Frobenius maps* $F : H^i_\mathfrak{m}(R) \to H^i_\mathfrak{m}(R)$ *are injective for all* $i \in \mathbb{Z}$*.*
- F*-full if* $H^i_\mathfrak{m}(R)$ *is a full* $R[F]$*-module for all* $i \in \mathbb{Z}$*.*
- F*-anti-nilpotent if* $H^i_\mathfrak{m}(R)$ *is an anti-nilpotent* $R[F]$*-module for all* $i \in \mathbb{Z}$*.*

[1] In this article we will mostly deal with left $R[F]$-modules. Whenever not specified, an $R[F]$-module will always mean a left $R[F]$-module.

By the discussions above, F-anti-nilpotent local rings are both F-full and F-injective. It is easy to see that F-full rings need not be F-injective; in fact, any Cohen-Macaulay local ring is F-full but not necessarily F-injective (e.g., F-injective rings are necessarily reduced, see [24, Lemma 3.11]). Conversely, there are examples of F-injective rings which are not F-full [23, Example 3.5]. In relation with the F-singularities defined above, we recall that F-pure local rings are F-anti-nilpotent by [20, Theorem 1.1], but the converse does not hold in general, see [24, Sections 5 and 6].

While the notions of being F-injective and F-full localize, see [6, Proposition 3.3] and [22, Proposition 2.7], it is not known if the same is true for the F-anti-nilpotent property:

Remark 2.6. *It is not known whether* F*-anti-nilpotency localizes. This localization property (and, in fact, a slightly stronger statement) was claimed in* [20, Theorem 4.6 and Theorem 5.10]*. However, the proof of* [20, Theorem 4.6 $(2) \Rightarrow (1)$] *contains a gap: the specific error in the argument was that, after localization, the completeness assumption was lost so one cannot use* [20, Theorem 4.4]*. In the* F*-finite local case, the Matlis dual of each Cartier submodule of* $H^{-i}(\omega_R^\bullet)$ *is an* F*-stable submodule of* $H^i_{\mathfrak{m}}(R)$*. Thus, the implication* [20, Theorem 4.6 $(1) \Rightarrow (2)$] *still holds, and the rest of the results in* [20, Section 5] *(except* [20, Theorem 5.10]*) are unaffected by this mistake.*

The gap in [20, Theorem 4.6] was first pointed out to the second author by Karl Schwede, who proposed to use the notion of Cartier-anti-nilpotency instead of F-anti-nilpotency to reconcile the localization property.

Definition 2.7. *An F-finite ring* R *of prime characteristic* p *is called Cartier-anti-nilpotent if* $H^{-i}(\omega_R^\bullet)$ *is an anti-nilpotent Cartier module (in the sense of* [26, Definition 1.19 on page 491]*) for each* i*, where* $\omega_R^\bullet$ *is the dualizing complex of* R*.*

We record some basic facts about Cartier-anti-nilpotency, see [26, Lemma 1.20 and Lemma 1.21 on page 491-492].

- For an F-finite local ring $(R, \mathfrak{m})$, if R is F-anti-nilpotent, then R is Cartier-anti-nilpotent.
- An F-finite ring R is Cartier-anti-nilpotent if and only if all localizations of R are Cartier-anti-nilpotent.

The following question is open; an affirmative answer would imply that F-anti-nilpotency localizes. See [26, Question 1.27 on page 495] for related questions.

Question 2.8. *Let* $(R, \mathfrak{m})$ *be an* F*-finite local ring of prime characteristic* p*. If* R *is Cartier-anti-nilpotent, then is it* F*-anti-nilpotent?*

In Proposition 2.10 we will prove that, in the graded situation, the above question has an affirmative answer. We need the following graded version of [18, Theorem 4.7] which is well-known to experts. We include a proof here for completeness using the theory of (graded) $\mathcal{F}$-modules, see [18] and [19] for their definitions and basic properties.

Proposition 2.9. *Let* R *be an* $\mathbb{N}$*-graded algebra over a field* k *of prime characteristic* p*. For any graded Artinian* $R[F]$*-module* W*, there exist a filtration of graded* $R[F]$*-modules*

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_n \subseteq N_n = W$$

such that each L_i/N_{i-1} *is a simple graded* $R[F]$*-module with nontrivial Frobenius action, and each* N_i/L_i *is a graded* $R[F]$*-module with nilpotent Frobenius action.*

Proof. First of all, we set $L_n := \langle F^e(W)\rangle$ for $e \gg 0$. Then L_n is graded and full, and W/L_n is nilpotent by construction. We next take any $R[F]$-submodule $N_{n-1} \subseteq L_n$ so that L_n/N_{n-1} is a simple $R[F]$-module, the existence of N_{n-1} is guaranteed by [18, Theorem 4.7] and the Frobenius action on L_n/N_{n-1} is necessarily nontrivial by the fullness of L_n (in particular L_n/N_{n-1} is full). Note that, a priori, we do not know whether N_{n-1} is graded, but we will show that we can always choose a graded N_{n-1}. To see this, write $R = S/I$ where S is an $\mathbb{N}$-graded polynomial ring over k and let $(-)^\vee$ be the graded Matlis duality, we have the following commutative diagram:

$$\begin{tikzcd}
(L_n/N_{n-1})^\vee \arrow[r, hook] \arrow[d, hook] & \mathcal{F}_S((L_n/N_{n-1})^\vee) \arrow[r, hook] \arrow[d, hook] & \mathcal{F}_S^2((L_n/N_{n-1})^\vee) \arrow[r, hook] \arrow[d, hook] & \cdots \\
L_n^\vee \arrow[r, hook] & \mathcal{F}_S(L_n^\vee) \arrow[r, hook] & \mathcal{F}_S^2(L_n^\vee) \arrow[r, hook] & \cdots
\end{tikzcd}$$

where the horizontal maps are injective by the fullness of L_n and L_n/N_{n-1}. After taking diret limit we obtain an inclusion of $\mathcal{F}_S$-modules:

$$\mathcal{H}(L_n/N_{n-1}) \hookrightarrow \mathcal{H}(L_n).$$

Since L_n is graded, $\mathcal{H}(L_n)$ is a graded $\mathcal{F}_S$-module. This implies that $\mathcal{H}(L_n/N_{n-1})$ is a graded $\mathcal{F}_S$-module by [19, Proposition 2.2]. In particular, we can choose a root M of $\mathcal{H}(L_n/N_{n-1})$ that is graded and the map $M \to \mathcal{H}(L_n/N_{n-1})$ is degree-preserving, see [19, discussion after Proposition 2.3]. Replacing M by $M \cap L_n^\vee$ we may assume that M is a graded submodule of $L_n^\vee$: it is still a root of $\mathcal{H}(L_n/N_{n-1})$ by simpleness of $\mathcal{H}(L_n/N_{n-1})$, which in turn implies that $M^\vee$ is a simple $R[F]$-module. Replacing N_{n-1} by $(L_n^\vee/M)^\vee$, we have that N_{n-1} is graded and $L_n/N_{n-1} = M^\vee$ is a simple $R[F]$-module. We can now continue this procedure: set $L_{n-1} = \langle F^e(N_{n-1})\rangle$ for $e \gg 0$ and choose a graded $R[F]$-submodule N_{n-2} of L_{n-1} so that L_{n-1}/N_{n-2} is a simple $R[F]$-module, etc. This process must terminate after finitely many steps by [18, Theorem 4.7]. Thus we obtain the desired filtration of graded $R[F]$-modules. □

Proposition 2.10. *Let R be an $\mathbb{N}$-graded algebra over an F-finite field of prime characteristic p. Then the ideal defining the non-Cartier-anti-nilpotent locus of R is homogeneous.*

Moreover, if $\mathfrak{m}$ is the unique maximal homogeneous ideal of R, the following conditions are equivalent:

(1) R is Cartier-anti-nilpotent;
(2) $R_\mathfrak{m}$ is Cartier-anti-nilpotent;
(3) $R_\mathfrak{m}$ is F-anti-nilpotent.

Proof. By Proposition 2.9 applied to the graded $R[F]$-modules $H^i_\mathfrak{m}(R)$, we obtain filtrations of graded $R[F]$-modules:

$$0 = L^i_0 \subseteq N^i_0 \subseteq L^i_1 \subseteq N^i_1 \subseteq \cdots \subseteq L^i_{n_i} \subseteq N^i_{n_i} = H^i_\mathfrak{m}(R)$$

with N^i_j/L^i_j nilpotent and L^i_j/N^i_{j-1} simple graded $R[F]$-modules. Applying graded Matlis duality yields filtrations of graded Cartier submodules of $H^{-i}(\omega_R^\bullet)$:

$$0 = C^i_0 \subseteq D^i_0 \subseteq C^i_1 \subseteq D^i_1 \subseteq \cdots \subseteq C^i_{n_i} \subseteq D^i_{n_i} = H^{-i}(\omega_R^\bullet).$$

where D^i_j/C^i_j are Cartier-nilpotent. Since $(\omega^\bullet_R)_P$, appropriately normalized, is a normalized dualizing complex of R_P for every $P \in \operatorname{Spec}(R)$. We obtain that the non-Cartier-anti-nilpotent locus is precisely

$$\bigcup_{i,j} \operatorname{Supp}_R(D^i_j/C^i_j).$$

In particular, its defining ideal is homogeneous since all C^i_j, D^i_j are finitely generated graded modules. For the last assertion, $(1) \Leftrightarrow (2)$ follows immediately from the already established statement, and $(3) \Rightarrow (2)$ always holds. To see $(2) \Rightarrow (3)$, simply notice that if $D^i_j/C^i_j = 0$ for all i, j then $N^i_j/L^i_j = 0$ by graded Matlis duality, thus R_m is F-anti-nilpotent. $\square$

Remark 2.11. *In Proposition 2.10, we do not know whether* $(1) - (3)$ *implies that* R_P *is* F*-anti-nilpotent for all* $P \in \operatorname{Spec}(R)$*. The same issue in establishing the localization property of* F*-anti-nilpotency still occurs: after localization we lost the graded assumption so it is not clear that every* F*-stable submodule of the local cohomology of* R_P *arises from the Matlis dual of a Cartier submodule of the cohomology of* $\omega^\bullet_{R_P}$.

2.3. **Herzog ideals and F-anti-nilpotency.** We can now prove the first result of this paper, namely that homogeneous Herzog ideals define F-anti-nilpotent quotients after localizing at the homogeneous maximal ideal.

Theorem 2.12. *Let* R *be an* $\mathbb{N}$*-graded algebra over an* F*-finite field* k *of prime characteristic* p *with homogeneous maximal ideal* $\mathfrak{m}$*. Write* $R = S/I$ *where* S *is an* $\mathbb{N}$*-graded polynomial ring over* k*. If* I *is a Herzog ideal, then* $R_\mathfrak{m}$ *is* F*-anti-nilpotent.*

Proof. Let us choose a monomial order on S so that $\operatorname{in}(I)$ is squarefree. By [8, 15.3], we can find a homogeneous ideal $\widetilde{I} \subseteq S[t]$ so that

$$S[t]/(\widetilde{I}, t) \cong S/\operatorname{in}(I) \text{ and } (S[t]/\widetilde{I}) \otimes_{k[t]} k(t) \cong R \otimes_k k(t).$$

Since $\operatorname{in}(I)$ is squarefree, $S/\operatorname{in}(I)$ is F-pure and thus $(S/\operatorname{in}(I))_\mathfrak{m}$ is F-anti-nilpotent by [20, Theorem 3.8]. It follows from [22, Theorem 4.2] that $(S[t]/\widetilde{I})_{\mathfrak{m}+(t)}$ is F-anti-nilpotent and thus Cartier-anti-nilpotent. Hence $S[t]/\widetilde{I}$ is Cartier-anti-nilpotent by Proposition 2.10. But then $(S[t]/\widetilde{I}) \otimes_{k[t]} k(t) \cong R \otimes_k k(t)$ is Cartier-anti-nilpotent as this property is stable under localization. It follows that $(R \otimes_k k(t))_\mathfrak{m}$ is F-anti-nilpotent by Proposition 2.10. Since $H^i_\mathfrak{m}(R_\mathfrak{m}) \otimes_k k(t) \cong H^i_\mathfrak{m}((R \otimes_k k(t))_\mathfrak{m})$, it is easy to see that the Frobenius action is anti-nilpotent on $H^i_\mathfrak{m}(R_\mathfrak{m})$ (as this is true after base change along $k \to k(t)$), thus $R_\mathfrak{m}$ is F-anti-nilpotent. $\square$

Corollary 2.13. *Let* R *be an* $\mathbb{N}$*-graded Algebra with Straightening Law (ASL) over a field* k *of prime characteristic* p*. Then* $R_\mathfrak{m}$ *is* F*-anti-nilpotent, where* $\mathfrak{m} = R_+$.

Proof. We have that $R = S/I$ where S is an $\mathbb{N}$-graded polynomial ring over k, and $\operatorname{in}(I)$ is squarefree for any DegRevLex monomial order extending the partial order on the variables given by the poset governing the ASL structure of R. So the result follows by Theoerem 2.12. $\square$

3. Herzog ideals versus F-purity for all $p \gg 0$

Throughout this section any polynomial ring $S = A[X_0. \dots, X_n]$ over a commutative ring A will be standard graded, i.e., $A = S_0$ and $\deg(X_i) = 1$ for all $i = 0, \dots, n$. In this section we study Question 1.2. We first investigate the case of nonsingular curves.

Proposition 3.1. *Let* $I \subseteq \mathbb{Q}[X_0, \dots, X_n]$ *be a homogeneous ideal such that* $I_\mathbb{C}$ *defines a projective (connected) nonsingular curve* $C \subseteq \mathbb{P}^n_\mathbb{C}$ *containing at least one* $\mathbb{Q}$*-rational point* $P \in C$*. Then the following conditions are equivalent:*

(1) $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ *is* F*-pure for all primes* $p \gg 0$*.*

(2) Possibly after a change of variables, $I_\mathbb{C} \subseteq \mathbb{C}[X_0, \dots, X_n]$ *is a Herzog ideal.*

Proof. (2) $\implies$ (1). Since $C \subseteq \mathbb{P}^n$ is nonsingular, it must be a rational normal curve by [14], so (1) follows because $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ is, for $p \gg 0$, a direct summand of $\overline{\mathbb{F}_p}[X^iY^{n-i} : 0 \leq i \leq j \leq n]$, which is in turn a direct summand of $\overline{\mathbb{F}_p}[X, Y]$ that is obviously F-pure.

(1) $\implies$ (2). Our assumption guarantees that C has genus at most 1 (see [1, Remark 1.2.10]) and the embedding $C \subseteq \mathbb{P}^n$ must be projectively normal. In fact, it is well-known that $H^1_\mathfrak{m}(R)_{\leq 0} = 0$, for example see [5, Proof of Theorem 5.9], while (1) guarantees that $H^1_\mathfrak{m}(R)_{>0} = 0$. Assume by contradiction that C has genus 1. Since it has a $\mathbb{Q}$-rational point, C is then an elliptic curve over $\mathbb{Q}$. By [9], there are then infinitely many supersingular primes p, and the homogeneous coordinate ring $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ is thus not F-pure for infinitely many primes p by [1, Remark 1.3.9]. This contradicts (1). So C has genus 0, and therefore $C \subseteq \mathbb{P}^n$ is a rational normal curve of degree $d \leq n$. So after a change of variables $I_\mathbb{C}$ is the ideal of 2-minors of the matrix

$$U = \begin{pmatrix} X_0 & X_1 & \dots & X_{d-1} \\ X_1 & X_2 & \dots & X_d \end{pmatrix},$$

and $\mathrm{in}(I_2(U)) = (X_iX_j : 1 \leq i + 1 < j \leq d)$ for LEX $X_0 > X_1 > \dots > X_n$. □

Remark 3.2.
- *We do not know whether the implication "(2) $\implies$ (1)" in Question 1.2 holds when* $I_\mathbb{C}$ *defines a singular projective curve.*
- *On the other hand, "(2) $\implies$ (1)" does not hold already if* $I_\mathbb{C}$ *defines a projective surface. For example, the ideal*

$$I = (XY, XZ, Y(ZU - W^2)) \subseteq \mathbb{Q}[X, Y, Z, U, W]$$

is such that $R_p = \mathbb{F}_p[X, Y, Z, U, W]/I_p$ *is not* F*-pure for any prime* p*, however we have* $\mathrm{in}(I_\mathbb{C}) = (XY, XZ, YZU)$ *for LEX with* $X > Y > Z > U > W$*. To see that* R_p *is not* F*-pure one can argue as follows: if* $u = \overline{U} \in R_p$*, the localization* $(R_p)_u$ *is isomorphic to* $A_p[U, U^{-1}]$ *where* $A_p = \mathbb{F}_p[X, Y, Z, W]/(XY, XZ, Y(Z - W^2))$ *(e.g., using* [2, Proposition 1.5.18]*). If* R_p *is* F*-pure so is* A_p*; however,* A_p *is not* F*-pure for any prime integer* p *by* [27, Example 3.2].
- *It is well-known that "(2) $\implies$ (1)" in Question 1.2 holds when* $\mathbb{C}[X_0, \dots, X_n]/I_\mathbb{C}$ *is Gorenstein. Indeed, if* $I_\mathbb{C} \subseteq \mathbb{C}[X_0, \dots, X_n]$ *is a Herzog ideal after a change of coordinates, for all* $p \gg 0$ *there exists a field* k *of characteristic* p *and a change of variables so that* $I_k \subseteq k[X_0, \dots, X_n]$ *becomes a Herzog ideal. In particular,* $k[X_0, \dots, X_n]/I_k$ *is* F*-injective. However, for* $p \gg 0$ *we have that* $k[X_0, \dots, X_n]/I_k$ *is also Gorenstein, and hence* F*-pure. Therefore* $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ *is* F*-pure as well.*

We are not aware of any counterexample to "(1) $\implies$ (2)" in Question 1.2. Note that, if $f \in \mathbb{Q}[X_0, \dots, X_n]$ is a homogeneous polynomial of degree $d \leq n$ defining a nonsingular hypersurface $X \subseteq \mathbb{P}^n_\mathbb{C}$, then $\mathbb{F}_p[X_0, \dots, X_n]/(\overline{f})$ is F-pure for all $p \gg 0$ by [10, Theorem 2.5]. As a first example, consider $f = X^3 + Y^3 + Z^3 + W^3 \in \mathbb{Q}[X, Y, Z, W]$ and $I = (f)$. Then $I_p \subseteq \mathbb{F}_p[X, Y, Z, W]/(X^3 + Y^3 + Z^3 + W^3)$ defines an F-pure quotient for all primes $p > 3$.

In this case, if $g \in \mathrm{Aut}(\mathbb{C}[X,Y,Z,W])$ is defined as $g(X) = X+Y+Z$, $g(Y) = -(X+Y)$, $g(Z) = -(X+Z)$, and $g(W) = X+W$, then it turns out that

$$g(f) = 6XYZ + 3Y^2Z + 3YZ^2 + 3X^2W + 3XW^2 + W^3,$$

so $\mathrm{in}(g(f)) = XYZ$ with respect to DegRevLex with $X > Y > Z > W$.

We next show that this is indeed a property common to all klt cubic hypersurfaces.

Theorem 3.3. *Let* $f \in \mathbb{Q}[X_0, \dots, X_n]$ *be a homogeneous polynomial of degree* $d \leq 3$*. Set* $I = (f)$ *and* $H = \mathrm{Proj}\,\mathbb{C}[X_0, \dots, X_n]/I_{\mathbb{C}}$*. Assume either* H *has klt singularities*[2] *or* H *is a curve, then the following conditions are equivalent:*

(1) $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ *is* F*-pure for all primes* $p \gg 0$*.*

(2) Possibly after a change of variables, $I_{\mathbb{C}} \subseteq \mathbb{C}[X_0, \dots, X_n]$ *is a Herzog ideal.*

Proof. First of all, since $\mathbb{C}[X_0, \dots, X_n]/I_{\mathbb{C}}$ is a hypersurface and in particular Gorenstein, $(2) \implies (1)$ is a special case of the last item of Remark 3.2. We thus focus on $(1) \implies (2)$. If $d = 1$ the conclusion is trivial, and if $d = 2$ it immediately follows by the classification of quadrics. In what follows we assume that $d = 3$.

In the case that H is a curve, then [9, Theorem 2] (and the assumption (1)) guarantees that H is necessarily a singular projective curve in $\mathbb{P}^2_{\mathbb{C}}$. We can assume that $P = [1:0:0]$ is a singular point of H, that is, we can do a change of coordinates $\alpha \in \mathrm{Aut}(\mathbb{C}[X,Y,Z])$ such that $X^3 \notin \mathrm{supp}(\alpha(f))$ and

$$\frac{\partial(\alpha(f))}{\partial X}(P) = \frac{\partial(\alpha(f))}{\partial Y}(P) = \frac{\partial(\alpha(f))}{\partial Z}(P) = 0.$$

This means that $\alpha(f) = Xq + g$, where q is a quadric of $\mathbb{C}[Y,Z]$ and g is a cubic of $\mathbb{C}[Y,Z]$. As before, there exists a change of coordinates $\beta \in \mathrm{Aut}(\mathbb{C}[Y,Z])$ such that $\beta(q) = 0$, $\beta(q) = Y^2$ or $\beta(q) = YZ$: in the third case $\mathrm{in}(\beta \circ \alpha(f)) = XYZ$ for LEX with respect to $X > Y > Z$; in the second and first case, it is easy to see that f^{p-1} cannot contain $(XYZ)^{p-1}$ in its support thus by Fedder's criterion $\mathbb{F}_p[X,Y,Z]/(\bar{f})$ would not be F-pure and thus contradicting (1).

Now we assume that H has klt singularities, and we may assume that $n \geq 3$ (if $n = 2$ then H is a curve and we have already established this case).

First we assume $n = 3$: in this case $H \subseteq \mathbb{P}^3$ is a cubic surface. If H is nonsingular, then there exist linear forms $l_i, m_i \in S = \mathbb{C}[X_0, X_1, X_2, X_3]$ where $i = 1, 2, 3$ such that

$$f = l_1 l_2 l_3 - m_1 m_2 m_3.$$

This goes back to the 19th century and is known as Cayley-Salmon equation, see [12] for a modern treatment. Any triple of the linear forms l_i, m_i for $i = 1, 2, 3$ are linearly independent (see [12, Section 2.3]) and $(l_i, m_i : i = 1, 2, 3) = (X_0, X_1, X_2, X_3)$ (otherwise the point corresponding to $(l_i, m_i : i = 1, 2, 3)$ would be a singular point of H). Hence l_1, l_2, l_3 and one of the m_i, say m_1, are linearly independent, so there exists a change of variables $\phi \in \mathrm{Aut}(S)$ such that

$$\phi(f) = X_0 X_1 X_2 - X_3 \phi(m_2) \phi(m_3).$$

In particular $\mathrm{in}(\phi(f)) = X_0 X_1 X_2$ for DegRevLex with $X_0 > X_1 > X_2 > X_3$.

If H is a singular cubic surface, then the defining equations of such H are classified by [25, Table 2 and Theorem 2] (all but the last equation define klt hypersurface H). From

[2] This condition implies that $\mathbb{C}[X_0, \dots, X_n]/I_{\mathbb{C}}$ is klt when $n \geq 3$ by [17] or [28] (for hypersurfaces, klt is equivalent to rational singularities) and thus is F-pure for all primes $p \gg 0$ by [13, Theorem 5.2].

those explicit equations, it is clear that one can choose a monomial order so that the initial term is squarefree except the following three cases:

(1) $f = X_3X_0^2 + X_1^3 + X_2^3$
(2) $f = X_3X_0^2 + X_0X_2^2 + X_1^2X_2$
(3) $f = X_3X_0^2 + X_0X_2^2 + X_1^3$

We now tackle these three cases by hand. In case (1), we consider the change of variables $\phi \in \mathrm{Aut}(\mathbb{C}[X_0, X_1, X_2, X_3])$ such that $\phi(X_0) = X_0 + X_1$, $\phi(X_1) = -(X_1 + X_3)$, $\phi(X_2) = X_1 + X_2$, $\phi(X_3) = 3X_3$. A straightforward computation shows that

$$\phi(f) = 6X_0X_1X_3 + 3X_0^2X_3 - 3X_1X_3^2 - X_3^3 + 3X_1^2X_2 + 3X_1X_2^2 + X_2^3$$

and it is clear that $\mathrm{in}(\phi(f)) = X_0X_1X_3$ for DegRevLex with $X_1 > X_0 > X_3 > X_2$. One can find a similar change of variables in case (2) and case (3), but for these two cases, one can alternatively consider the following two equations:

(2') $f = X_3X_0^2 + X_0X_2^2 + X_1^2X_2 + X_0X_1X_2$
(3') $f = X_3X_0^2 + X_0X_2^2 + X_1^3 + X_0X_1X_2$

The equation in $(2')$ has a unique D_5-singularity while the equation in $(3')$ has a unique E_6-singularity. Therefore, the hypersurfaces defined by the equations in $(2')$ and $(3')$ must be isomorphic to the hypersurfaces defined by the equations in (2) and (3) respectively by [25, Theorem 2]. It is left to observe that $\mathrm{in}(f) = X_0X_1X_2$ for DegRevLex $X_0 > X_1 > X_2 > X_3$ in case $(2')$ and for the monomial order by first declaring DegRevLex with respect to X_3 and then using LEX with $X_0 > X_1 > X_2$.

Finally, if $n > 3$, by Bertini's theorem (see [16, Lemma 5.17]), there exists a hyperplane section $H' \subseteq \mathbb{P}^{n-1}$ of $H \subseteq \mathbb{P}^n$ so that H' still has klt singularities. After a change of variables $\alpha \in \mathrm{Aut}(\mathbb{C}[X_0, \dots, X_n])$, we can assume that the hyperplane is $\{X_n = 0\}$; in other words,

$$\alpha(f) = f' + X_n f''$$

where $f'' \in S$ and $f' \in S' = \mathbb{C}[X_0, \dots, X_{n-1}]$ is a homogeneous polynomial of degree 3 defining a klt hypersurface $H' \subseteq \mathbb{P}^{n-1}$. By induction on n, there exists a change of variables $\phi' \in \mathrm{Aut}(S')$ such that $\mathrm{in}(\phi'(f')) = X_0X_1X_2$ for some monomial order in $X_0, X_1, \dots, X_{n-1}$. Extending ϕ' to $\beta \in \mathrm{Aut}(S)$ by putting $\beta(X_n) = X_n$, and defining $\phi = \beta \circ \alpha \in \mathrm{Aut}(S)$, we have $\mathrm{in}(\phi(f)) = X_0X_1X_2$ where we extend the monomial order from S' to S via DegRevLex with respect to X_n. $\square$

Remark 3.4. *We do not know the implication "*$(1) \implies (2)$*" in Question 1.2 in general even when* $I = (f)$ *is a principal ideal defining a nonsingular hypersurface in* $\mathbb{P}^n$ *when* $\deg(f) > 3$*. Note that an affirmative answer in this case would imply that, if* $I_{\mathbb{C}}$ *defines a nonsingular Calabi-Yau hypersurface* $X \subseteq \mathbb{P}^n$*, then there are infinitely many primes* p *such that* $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ *is not* F*-pure: indeed, it is easy to check that if* $f \in \mathbb{C}[X_0, \dots, X_n]$ *is a homogeneous polynomial of degree* $n+1$ *so that the initial term of* f *is squarefree for some monomial order, then the hypersurface of* $\mathbb{P}^n$ *defined by* f *is singular.*

More generally, an affirmative answer to the implication "$(1) \implies (2)$*" in Question 1.2 in the nonsingular case, together with a solution of* [4, Conjecture 2] *would imply that, if* $I_{\mathbb{C}}$ *defines a nonsingular Calabi-Yau variety of* $X \subseteq \mathbb{P}^n$*, then there are infinitely many primes* p *such that* $\mathbb{F}_p[X_0, \dots, X_n]/I_p$ *is not* F*-pure.*

4. Annihilators of F-stable submodules and uniformly compatible ideals

In this section we prove some results regarding annihilators of F-stable submodules and subquotients of local cohomology modules. Our first result is a generalization of the main result of [20] to not necessarily F-pure rings. We recall that the trace ideal of an R-module M is the ideal $\sum_{\phi \in \mathrm{Hom}_R(M,R)} \phi(M) \subseteq R$.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p. Then there exists $e > 0$ so that the trace ideal J_e of $F^e_* R$ annihilates every F-stable subquotient of $H^i_{\mathfrak{m}}(R)$ that is nilpotent.*

In particular, if R is F-pure, then the Frobenius action on $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent, i.e., R is F-anti-nilpotent.

Proof. First of all, we take a Lyubeznik filtration of $H^i_{\mathfrak{m}}(R)$ (see [18, Theorem 4.7]):

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_n \subseteq N_n = H^i_{\mathfrak{m}}(R)$$

of F-stable submodules of $H^i_{\mathfrak{m}}(R)$ so that each L_i/N_{i-1} is a simple $R[F]$-module with non-trivial Frobenius action and each N_i/L_i is an $R[F]$-module with nilpotent Frobenius action. There exists e_0 so that $F^{e_0}(N_i/L_i) = 0$ for all i. Then, it follows by an easy filtration argument that for any F-stable subquotient N/L of $H^i_{\mathfrak{m}}(R)$ that is nilpotent, $F^{ne_0}(N/L) = 0$. Set $e = ne_0$. For any $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, we need to show that $\phi(F^e_* R)$ annihilates N/L. We may replace L by $L' := \langle F^{e'}(L) \rangle$ for $e' \gg 0$: the Frobenius action on L/L' is nilpotent and thus so is N/L', and clearly if $\phi(F^e_* R)$ annihilates N/L' then it also annihilates N/L. After this replacement, we have that L is full. We now consider the following commutative diagram:

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(R) \arrow[r] & F^e_* R \otimes H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[rr, bend left, "\phi\otimes \mathrm{id}"] & H^i_{\mathfrak{m}}(F^e_* R) \arrow[r, "H^i_{\mathfrak{m}}(\phi)"] & H^i_{\mathfrak{m}}(R) \\
N \arrow[r] \arrow[u, hook] & F^e_* R \otimes N \arrow[r] \arrow[u] & F^e_* N \arrow[u, hook] & \\
L \arrow[r] \arrow[u, hook] & F^e_* R \otimes L \arrow[r, two heads] \arrow[u] & F^e_* L \arrow[u, hook] &
\end{tikzcd}$$

where the top part is commutative by [20, Lemma 3.3] (see also [26, Lemma 1.23 on page 493]), and the bottom right surjectivity follows from the fact that L is full. By our choice of e, we know that

$$\mathrm{Image}(F^e_* R \otimes N \to F^e_* N) \subseteq F^e_* L = \mathrm{Image}(F^e_* R \otimes L \to F^e_* L).$$

Chasing the diagram, we found that

$$\mathrm{Image}(F^e_* R \otimes N \to H^i_{\mathfrak{m}}(R)) \subseteq \mathrm{Image}(F^e_* R \otimes L \to H^i_{\mathfrak{m}}(R)).$$

Thus by the commutative diagram (use the arrow $\phi \otimes \mathrm{id}$), we have that

$$\phi(F^e_* R) N \subseteq L$$

as wanted. $\square$

We next recall that for an F-finite ring R, an ideal $I \subseteq R$ is said *uniformly compatible* if $\phi(I) \subseteq I$ for all $\phi \in \mathrm{Hom}_R(F_* R, R)$.

Proposition 4.2. *Let* $(R, \mathfrak{m})$ *be an* F*-finite local ring of prime characteristic* p *and* $N \subseteq H^i_\mathfrak{m}(R)$ *be an* F*-stable submodule that is full. Then for any* F*-stable submodule* L *that contains* N*,* $J := \mathrm{Ann}_R(L/N)$ *is a uniformly compatible ideal.*

In particular, the annihilator of any F*-stable submodule* $L \subseteq H^i_\mathfrak{m}(R)$ *is a uniformly compatible ideal.*

Proof. Let $\phi \in \mathrm{Hom}_R(F_*R, R)$. We consider the following commutative diagram

$$\begin{tikzcd}
H^i_\mathfrak{m}(R) \arrow[r] & F_*R \otimes H^i_\mathfrak{m}(R) \arrow[r] \arrow[rr, bend left, "\phi\otimes \mathrm{id}"] & H^i_\mathfrak{m}(F_*R) \arrow[r, "H^i_\mathfrak{m}(\phi)"] & H^i_\mathfrak{m}(R) \\
L \arrow[r] \arrow[u, hook] & F_*R \otimes L \arrow[r] \arrow[u] & F_*L \arrow[u, hook] & \\
N \arrow[r] \arrow[u, hook] & F_*R \otimes N \arrow[r, two heads] \arrow[u] & F_*N \arrow[u, hook] &
\end{tikzcd}$$

where the top part is commutative by [20, Lemma 3.3] (see also [26, Lemma 1.23 on page 493]), and the bottom right surjectivity follows from the hypothesis that N is full.

We need to show $\phi(F_*J) \subseteq J$. Since $J = \mathrm{Ann}_R(L/N)$, it suffices to show that $\phi(F_*J)L \subseteq N$. But by the commutative diagram above, we have

$$\phi(F_*J)L \subseteq H^i_\mathfrak{m}(\phi)(F_*J \cdot F_*L) \subseteq H^i_\mathfrak{m}(\phi)(F_*N) = H^i_\mathfrak{m}(\phi)(\mathrm{Image}(F_*R \otimes N)) \subseteq \phi(F_*R)N \subseteq N.$$

Hence the first statement follows. The second statement follows from the first by applying it to the full submodule $N = 0$. ☐

Proposition 4.3. *Let* $(R, \mathfrak{m})$ *be an* F*-finite local ring of prime characteristic* p*. If the annihilator* J *of an* F*-stable subquotient of* $H^i_\mathfrak{m}(R)$ *is radical (e.g., if the Frobenius action on that subquotient is injective), then* J *is a uniformly compatible ideal.*

In particular, if R *is* F*-anti-nilpotent, then the annihilator of any* F*-stable subquotient of* $H^i_\mathfrak{m}(R)$ *is uniformly compatible.*

Proof. Let L/N be an F-stable subquotient of $H^i_\mathfrak{m}(R)$ so that $J := \mathrm{Ann}(L/N)$ is radical. Suppose J is not uniformly compatible. Then there is $\phi \in \mathrm{Hom}_R(F^e_*R, R)$ so that $\phi(F^e_*J) \nsubseteq J$. Since J is radical, there exists a minimal prime P of J so that $\phi(F^e_*J) \nsubseteq P$. Thus after localizing at P, $\phi(F^e_*J_P) = R_P$. It follows that R_P is F-pure and hence F-anti-nilpotent.

The inclusion of F-stable submodules $N \subseteq L \subseteq H^i_\mathfrak{m}(R)$ yields an inclusion of Cartier submodules $C \subseteq D \subseteq H^{-i}(\omega^\bullet_R)$, with $\mathrm{Ann}_R(D/C) = J$. After localizing at P, we obtain an inclusion of Cartier submodules $C_P \subseteq D_P \subseteq H^{-j}(\omega^\bullet_{R_P})$ with $J_P = \mathrm{Ann}_{R_P}(D_P/C_P)$, which in turn yields an inclusion of F-stable submodules $N' \subseteq L' \subseteq H^j_P(R_P)$ with $\mathrm{Ann}_{R_P}(L'/N') = J_P$. Since R_P is F-anti-nilpotent, every F-stable submodule of $H^j_P(R_P)$ is full by [22, Lemma 2.1] and thus by Proposition 4.2, J_P is uniformly compatible in R_P, which contradicts $\phi(F^e_*J_P) = R_P$. ☐

Based on Proposition 4.2 and Proposition 4.3, one might ask whether the annihilator of every F-stable subquotient of $H^i_\mathfrak{m}(R)$ is uniformly compatible. We include a simple example indicating that this is not the case.

Example 4.4. *Let* $R = S/I$ *with* $S = \mathbb{F}_2[[x,y]]$ *and* $I = (x,y)^2$*, and let* $\mathfrak{m} = (x,y)$*. If we set* $N = (x)$*, then* N *is an* F*-stable submodule of* $L = R = H^0_{\mathfrak{m}}(R)$*, and*

$$\mathrm{Ann}_R(L/N) = \mathrm{Ann}_R(R/(x)) = (x).$$

Now for $q = 2^e$*, we have* $f_e := x^{q-2}y^{2q-1} \in I^{[q]} :_S I$*. Let* $\mathrm{Tr} : F_*(S) \to S$ *denote the trace map. We consider* $\phi_e(-) = \mathrm{Tr}^e(f_e \cdot -) : F^e_*(R) \to R$ *and it is easy to see that*

$$\phi_e(x) = \mathrm{Tr}^e(x^{q-1}y^{2q-1}) = y\mathrm{Tr}^e(x^{q-1}y^{q-1}) = y \notin (x),$$

thus (x) *is not uniformly compatible.*

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