

GRÖBNER DEFORMATIONS AND F -SINGULARITIES

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ABSTRACT. For polynomial ideals in positive characteristic, defining F -split rings and admitting a squarefree monomial initial ideal are different notions. In this note we show that, however, there are strong interactions in both directions. Moreover we provide an overview on which F -singularities are Gröbner deforming. Also, we prove the following characteristic-free statement: if \mathfrak{p} is a height h prime ideal such that $\text{in}(\mathfrak{p}^{(h)})$ contains at least one squarefree monomial, then $\text{in}(\mathfrak{p})$ is a squarefree monomial ideal.

1. INTRODUCTION

The motivation for this note has been, essentially, trying to achieve a better understanding of the following question:

Question 1.1. *Let I be an ideal of a polynomial ring S over a field K . When is there a monomial order $<$ on S such that $\text{in}_{<}(I)$ is squarefree?*

One of the reasons why this is an interesting problem arises from the recent work [10] by Conca and the second author of this paper, roughly stating that I and $\text{in}_{<}(I)$ are much more related than usual provided the latter is a squarefree monomial ideal. There are already many known classes of ideals I (and suitable monomial orders) such that $\text{in}_{<}(I)$ is squarefree, such as ideals defining Algebras with Straightening Law, Cartwright-Sturmfels ideals and Knutson ideals. In Theorem 3.13 we identify a new class: If I is a radical ideal, as soon as $\text{in}_{<}(I^{(h)})$ contains a squarefree monomial, where h is the maximum height of a minimal prime ideal of I , then $\text{in}_{<}(I)$ is a squarefree monomial ideal. The result is proved first in positive characteristic, and then derived over fields of characteristic 0. The proof in positive characteristic relies on the “ F -split” notion and a suitable version of Fedder’s criterion, see Theorem 3.12.

If K has positive characteristic, in which case we can speak of F -singularities (where F stands for the Frobenius endomorphism), we investigate the following:

Question 1.2. *For which kind of F -singularities do we have that S/I has those F -singularities provided that, for some weight vector $w \in \mathbb{N}^n$, $S/\text{in}_w(I)$ has those F -singularities?*

The two questions above are actually related: if $\text{in}_{<}(I)$ is squarefree, then $S/\text{in}_{<}(I)$ is F -split. Although there are examples of ideals I such that $\text{in}_{<}(I)$ is squarefree but S/I is not F -split (e.g. see Example 4.13), it turns out that S/I is always F -injective, and even strongly F -injective, provided $\text{in}_{<}(I)$ is squarefree (see Corollary 4.11).

On the other hand, it is very easy to find examples of ideals I such that S/I is F -split but $\text{in}_{<}(I)$ is not squarefree for any monomial order. However, Theorem 3.12 states that many ideals defining F -split rings admit a squarefree initial ideal; hence, to some extent, “being F -split” and “admitting a squarefree initial ideal” are connected properties.

Question 1.2 is also related to the so-called deformation problem: if (R, \mathfrak{m}) is a Noetherian local ring and $x \in \mathfrak{m}$ is a nonzero divisor on R such that R/xR has some property \mathcal{P} , is it true

that R has property \mathcal{P} as well? Of course the answer depends on the property \mathcal{P} , for example it is positive if \mathcal{P} is “being a domain” and negative if \mathcal{P} is “being irreducible”. There is fervent research on the deformation problem when \mathcal{P} is an F -singularity, especially if “ $\mathcal{P} = F$ -injective”, in which case the problem is still open. Regarding Question 1.2, the answers we get agree with the answers to the deformation problem; this is expected, though it needs some explanations. On the way we characterize in Lemma 2.1 the deformations arising as Gröbner deformations.

2. GRÖBNER DEFORMATIONS

Throughout this note, by a ring we mean a Noetherian commutative ring with unity. A \mathbb{N} -graded ring is a ring $R = \bigoplus_{i \in \mathbb{N}} R_i$ (usually R_0 will be a field). A \mathbb{N} -graded ring $R = \bigoplus_{i \in \mathbb{N}} R_i$ is standard graded if $R = R_0[R_1]$.

Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K and $I \subset S$ be an ideal. If $<$ is a monomial order on S we can consider the initial ideal $\text{in}_<(I) \subset S$ generated by all the monomials of the form $\text{in}_<(f)$ with $f \in I$. It turns out that it is possible to choose a suitable weight vector $w \in (\mathbb{N}_{>0})^n$ (depending on $<$ and I) such that $\text{in}_<(I) = \text{in}_w(I)$. Here $\text{in}_w(I)$ is the ideal of S generated by $\text{in}_w(f)$ with $f \in I$, where $\text{in}_w(f)$ stands for the sum of the terms of f with maximal w -degree. The latter point of view is more convenient concerning some aspects. For example, besides Gröbner bases it also includes Sagbi bases. In fact, if $A \subset S$ is a K -subalgebra of S , consider the K -subalgebra $\text{in}_<(A) \subset S$ generated by all the monomials of the form $\text{in}_<(f)$ with $f \in A$. If $f_1, \dots, f_m \in A$ are a Sagbi basis of A , that is $\text{in}_<(A) = K[\text{in}_<(f_1), \dots, \text{in}_<(f_m)]$, it is easy to see that $A = K[f_1, \dots, f_m]$. It turns out that, if $J \subset P = K[Y_1, \dots, Y_m]$ is the kernel of the K -algebra homomorphism sending Y_i to f_i (so that $P/J \cong A$), there exists $u \in (\mathbb{N}_{>0})^m$ such that $\text{in}_u(J)$ is the kernel of the K -algebra homomorphism sending Y_i to $\text{in}_<(f_i)$, hence $\text{in}_u(J)$ is a binomial ideal and $P/\text{in}_u(J) \cong \text{in}_<(A)$, (cf. [9, Corollary 2.1]).

The formation of $\text{in}_w(I)$ can also be seen as a deformation: Let t be an extra homogenizing variable, and $\text{hom}_w(I) \subset S[t]$ the w -homogenization of I . Then we say that $R = S[t]/\text{hom}_w(I)$ is a *Gröbner deformation*, (see Chapter 15 of [16]) and we have that:

- R is a \mathbb{N} -graded ring such that $R_0 = K$ and $t \cdot 1 \in R$ has degree 1 (the grading is given by $\deg(X_i \cdot 1) = w_i$ and $\deg(t \cdot 1) = 1$).
- t is a nonzero-divisor on R .
- $R/tR \cong S/\text{in}_w(I)$.
- $R/(t-1)R \cong S/I$.

A *1-parameter affine deformation over K* is a flat morphism $X \rightarrow \mathbb{A}^1$ where \mathbb{A}^1 is the affine line over K and X is an affine variety over K . In other words, a 1-parameter affine deformation over K is a K -algebra R which is a flat $K[t]$ -module (equivalently a $K[t]$ -module without nontrivial torsion). In the following we will write t for $t \cdot 1 \in R$.

Lemma 2.1. *Let R be a 1-parameter affine deformation over K . Then the following are equivalent:*

- (1) R is \mathbb{N} -graded, $R_0 = K$ and $t \in R$ is homogeneous of degree 1.
- (2) R is a Gröbner deformation.

Proof. We already noticed (2) \implies (1). For the converse, let $V \subset R \setminus K$ be a finite dimensional graded vector space containing t such that $R = K[V]$. Set $n+1 = \dim_K V$. Let $v_1, \dots, v_n \in V$ be homogeneous elements such that t, v_1, \dots, v_n is a K -basis of V , $S = K[X_1, \dots, X_n]$ and $S[z] \rightarrow R$ the K -algebra homomorphism sending z to t and X_i to v_i . Call $J \subset S[z]$ the kernel, $I = (J + (z-1))/(z-1) \subseteq S$, $w_i = \deg(v_i) = \deg(X_i)$ and put $\deg(z) = \deg(t) = 1$, so that the above map is graded. We claim that $J = \text{hom}_w(I)$ (so that $R/tR \cong S/\text{in}_w(I)$). So we would conclude because $R \cong S[z]/\text{hom}_w(I)$.

To prove the claim, it is useful to introduce the dehomogenization homomorphism $\pi : S[z] \rightarrow S$ sending X_i to itself and z to 1. With this notation $I = \pi(J)$.

Let us first see that $J \subset \text{hom}_w(I)$. Let F be a homogeneous element of J . We can write $F = z^r G$ where G is a homogeneous polynomial of $S[z]$ not divided by z . Of course $\pi(F) = \pi(G)$ belongs to I , so $\text{hom}_w(\pi(G)) = G \in \text{hom}_w(I)$. Since F is a multiple of G , it belongs to $\text{hom}_w(I)$ as well. Since J is a homogeneous ideal we conclude that $J \subset \text{hom}_w(I)$.

For the inclusion $\text{hom}_w(I) \subset J$, take $f \in I$ and consider $\text{hom}_w(f) \in \text{hom}_w(I)$. By definition $f = \pi(F)$ for some $F \in J$. Since J is homogeneous, $F = \sum_i F_i$ where $F_i \in J$ is homogeneous of degree i . If $d = \max\{i : F_i \neq 0\}$, we can replace F with $F' = \sum_i z^{d-i} F_i$, which is a homogeneous element of J such that $\pi(F') = f$. So we can assume at once that F is homogeneous. As before, we can write $F = z^r G$ where G is a homogeneous polynomial of $S[z]$ not divided by z . Since R is flat over $K[t]$, t is a nonzero-divisor on R , so that G belongs to J . So $\text{hom}_w(f) = G$ belongs to J . Since $\text{hom}_w(I)$ is generated by elements of the form $\text{hom}_w(f)$ with $f \in I$, we conclude that $\text{hom}_w(I) \subset J$. \square

In view of the previous lemma, we will refer to a \mathbb{N} -graded ring R which is a $K[t]$ -module without nontrivial torsion, such that $t \in R$ is homogeneous of degree 1 and such that $R_0 = K$, as a Gröbner deformation. We introduce the following concept:

Definition 2.2. *Let \mathcal{P} be some property that can have a ring. We say that \mathcal{P} is G -deforming if the following two conditions hold:*

- (1) *If R is a \mathbb{N} -graded ring with $R_0 = K$ and $x \in R$ is a nonzero-divisor on R of degree 1 such that R/xR has property \mathcal{P} , then R_x has property \mathcal{P} as well.*
- (2) *If R is a (not necessarily graded) ring such that $R[X, X^{-1}]$ has property \mathcal{P} , then R has property \mathcal{P} as well.*

Proposition 2.3. *Let R be a Gröbner deformation. If R/tR , has a G -deforming property \mathcal{P} , then $R/(t - \lambda)R$ has property \mathcal{P} as well for each $\lambda \in K$.*

In other words, if $I \subset S = K[X_1, \dots, X_n]$ is an ideal and $w \in \mathbb{N}^n$ a weight vector such that $S/\text{in}_w(I)$ has a G -deforming property \mathcal{P} , then S/I has property \mathcal{P} as well.

Proof. Since \mathcal{P} is G -deforming, then:

- Because R/tR has property \mathcal{P} , then R_t has property \mathcal{P} as well.
- Notice that, if $A = R/(t - 1)R$, $R_t \cong A[X, X^{-1}]$. Since $A[X, X^{-1}]$ has property \mathcal{P} , $A = R/(t - 1)R$ has property \mathcal{P} as well.

So $R/(t - 1)R$ has property \mathcal{P} . Now simply notice that, since R is a Gröbner deformation, $R/(t - \lambda)R$ is isomorphic to $R/(t - 1)R$ for $\lambda \in K \setminus \{0\}$. \square

Example 2.4. *The conclusion of Proposition 2.3 may fail for 1-parameter affine deformations over K which are not Gröbner. For example, if K is a field of characteristic 5, consider*

$$R = K[X, Y, Z, t]/(tX^3 + tY^3 + tZ^3 + XYZ).$$

such an R is a 1-parameter affine deformation over K and $R/tR \cong K[X, Y, Z]/(XYZ)$ is strongly F -injective. As we will see, being strongly F -injective is a G -deforming property, however $R/(t - 1)R \cong K[X, Y, Z]/(X^3 + Y^3 + Z^3 + XYZ)$ is not even F -injective.

3. F -SINGULARITIES AND GRÖBNER DEFORMATIONS

Let p be a prime number. Let R be a ring of characteristic p , and consider the *Frobenius map*:

$$\begin{aligned} F : R &\longrightarrow R \\ r &\longmapsto r^p \end{aligned}$$

Note that F is a ring homomorphism. We denote by F_*R the R -module defined as follows:

- $F_*R = R$ as additive group;
- $r \cdot x = r^p x$ for all $r \in R$ and $x \in F_*R$.

This way we can also think of F as the following map of R -modules:

$$\begin{aligned} F : R &\longrightarrow F_*R \\ r &\longmapsto r^p \end{aligned}$$

The ring R is reduced if and only if F is injective, so it is natural to introduce the following concept:

Definition 3.1. *R is F -split if there exists a homomorphism $\theta : F_*R \rightarrow R$ of R -modules such that $\theta \circ F = 1_R$. Such a θ is called an F -splitting of R .*

If I is an ideal of R , we have an induced map of R -modules $F : H_I^i(R) \rightarrow H_I^i(F_*R)$ for all $i \in \mathbb{N}$. As Abelian groups, it is easy to check that $H_{F(I)R}^i(R) = H_I^i(F_*R)$, hence, since $F(I)R = (x^p : x \in I)$ and I have the same radical, we have a map of Abelian groups:

$$F : H_I^i(R) \rightarrow H_I^i(R).$$

If R is F -split, of course $F : H_I^i(R) \rightarrow H_I^i(F_*R)$ splits as a map of R -modules. In particular, $F : H_I^i(R) \rightarrow H_I^i(R)$ is injective for any ideal $I \subset R$ and $i \in \mathbb{N}$. The latter fact turned out to be very powerful since the work of Hochster and Roberts [22]. Hence it has been natural to introduce the following definition:

Definition 3.2. *R is F -injective if the map $F : H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R)$ is injective for any maximal ideal $\mathfrak{m} \subset R$ and $i \in \mathbb{N}$.*

The “ F -split” property does not deform, i.e. there are examples of local rings R which are not F -split but such that R/xR is F -split for some regular element $x \in R$ (see Example 4.13). It is still an open problem whether the “ F -injective” property deforms. For this reason we further need to introduce the following property:

Definition 3.3. *R is F -full if the image of the map $F : H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R)$ generates $H_{\mathfrak{m}}^i(R)$ as R -module for any maximal ideal $\mathfrak{m} \subset R$ and $i \in \mathbb{N}$.*

It turns out that, if R is F -split, then it is F -full ([24, Theorem 3.7] and [25, Remark 2.4]). Moreover the “ F -full property” is known to deform ([25, Theorem 4.2]). Since there is no relationship between being F -full and being F -injective (any Cohen-Macaulay ring is F -full; on the other hand there exist F -injective rings that are not F -full, see [26, Example 3.5]), we introduce the last F -singularity of this paper:

Definition 3.4. *R is strongly F -injective if it is F -injective and F -full.*

By the previous discussion it follows that being strongly F -injective is a property in between the “ F -split” and the “ F -injective” properties. The important point for us is that the “strongly F -injective” property deforms by [25, Corollary 5.16].

3.1. F -splittings of the polynomial ring. In this subsection, we essentially combine parts of Fedder’s paper [18] with parts of the more recent paper of Knutson [23], in order to find interesting ideals having a squarefree Gröbner degeneration.

For this subsection, K will be a perfect field of prime characteristic p and $S = K[X_1, \dots, X_n]$ the polynomial ring in n variables over K . It is easy to see that F_*S is the free S -module generated by the monomials $X_1^{i_1} \cdots X_n^{i_n}$ with $i_j < p$ for all j . In particular, S is F -split. We want to describe all the F -splittings $\theta : F_*S \rightarrow S$, and more generally the elements of $\text{Hom}_S(F_*S, S)$. Of course the latter is a free S -module generated by the dual basis of $X_1^{i_1} \cdots X_n^{i_n}$ with $i_j < p$ for all j , say ϕ_{i_1, \dots, i_n} . But our purpose is to understand the structure of $\text{Hom}_S(F_*S, S)$ as an F_*S -module.

To this goal, let us introduce the fundamental element $\text{Tr} := \phi_{p-1, p-1, \dots, p-1} \in \text{Hom}_S(F_*S, S)$. We claim that $\text{Hom}_S(F_*S, S)$, as an F_*S -module, is generated by Tr . More precisely, the following is an isomorphism of F_*S -modules:

$$\begin{aligned} \Phi : F_*S &\rightarrow \text{Hom}_S(F_*S, S) \\ f &\mapsto f \star \text{Tr} : g \mapsto \text{Tr}(fg) \end{aligned}$$

The fact that Φ is an injective map of F_*S -modules is clear. For the surjectivity, just notice that, if i_1, \dots, i_n are natural numbers such that $i_j < p$ for all j , we have $\phi_{i_1, \dots, i_n} = X_1^{p-i_1-1} \dots X_n^{p-i_n-1} \star \text{Tr}$.

Remark 3.5. Notice that, given $f \in S$, $f \star \text{Tr}$ is an F -splitting of S if and only if the following two conditions hold:

- (1) $X_1^{p-1} \dots X_n^{p-1} \in \text{supp}(f)$ and its coefficient in f is 1.
- (2) If $X_1^{u_1} \dots X_n^{u_n} \in \text{supp}(f)$ and $u_1 \equiv \dots \equiv u_n \equiv -1 \pmod{p}$, then $u_i = p-1 \forall i$.

Definition 3.6. If $\theta : F_*S \rightarrow S$ is an F -splitting, we say that an ideal $I \subset S$ is compatibly split with respect to θ if $\theta(I) \subset I$.

Remark 3.7. Of course, if an ideal $I \subset S$ is compatibly split with respect to an F -splitting θ , then $\bar{\theta} : (F_*S)/I = F_*(S/I) \rightarrow S/I$ defines an F -splitting of S/I ; in particular S/I is F -split. Furthermore, in this case, $\theta(I) = I$ (indeed the inclusion $I \subset \alpha(I)$ holds true for any F -splitting $\alpha \in \text{Hom}_S(F_*S, S)$).

Proposition 3.8. The map $\theta = X_1^{p-1} \dots X_n^{p-1} \star \text{Tr} \in \text{Hom}_S(F_*S, S)$ is an F -splitting of S , and the compatibly split ideals with respect to θ are exactly the squarefree monomial ideals of S .

Proof. That θ is an F -splitting is clear, and it is easy to check that a squarefree monomial ideal is compatibly split with respect to θ .

Viceversa, let $g = \sum_{i=1}^s a_i \mu_i \in I$, where $\mu_i = X_1^{u_{i1}} \dots X_n^{u_{in}}$ and $a_i \in K \setminus \{0\}$. Pick $i \in \{1, \dots, s\}$. Our purpose is to show that, if I is a compatibly split ideal with respect to θ , then $\mu_i \in I$. Clearly, there exists $N \in \mathbb{N}$ such that, for all $i \neq k \in \{1, \dots, s\}$, $u_{kj} \not\equiv u_{ij} \pmod{p^N}$ for some $j \in \{1, \dots, n\}$. For each $j = 1, \dots, n$, let $0 \leq v_j < p^N$ such that $u_{ij} \equiv -v_j \pmod{p^N}$, and call $h = X_1^{v_1} \dots X_n^{v_n} g \in I$. Since I is a compatibly split ideal with respect to θ , then $\theta^N(h) \in I$. Notice that the monomials in the support of $\theta^N(h)$ correspond to those $k \in \{1, \dots, s\}$ such that $u_{kj} \equiv u_{ij} \pmod{p^N}$ for all $j = 1, \dots, n$. Hence $\theta^N(h)$ is a monomial, precisely

$$\theta^N(h) = \sqrt[p^N]{a_i X_1^{u_{i1}+v_1} \dots X_n^{u_{in}+v_n}}.$$

Since $\frac{u_{ij}+v_j}{p^N} \leq u_{ij}$ for any $j = 1, \dots, n$, μ_i is a multiple of $\theta^N(h) \in I$, so that $\mu_i \in I$. This shows that I is a monomial ideal. That I is radical follows from the fact that S/I is F -split. \square

The following proposition has already been proved in [23, Lemma 2]. We provide a proof here for the convenience of the reader.

Proposition 3.9. Let $w = (w_1, \dots, w_n) \in (\mathbb{N}_{>0})^n$ be a weight vector. Then, for any $g \in S$, either $\text{Tr}(\text{in}_w(g)) = 0$ or $\text{Tr}(\text{in}_w(g)) = \text{in}_w(\text{Tr}(g))$.

Proof. Given two vectors $(u_1, \dots, u_n), (v_1, \dots, v_n) \in \mathbb{R}^n$ clearly we have:

$$(3.1) \quad \sum_{i=1}^n u_i w_i \geq \sum_{i=1}^n v_i w_i \iff \sum_{i=1}^n \left(\frac{u_i+1}{p} - 1 \right) w_i \geq \sum_{i=1}^n \left(\frac{v_i+1}{p} - 1 \right) w_i.$$

Recall that, if $\mu = X_1^{u_1} \dots X_n^{u_n}$ is a monomial of S , $w(\mu) = \sum_{j=1}^n w_j u_j$, and that $w(f) = \max\{w(\nu) : \nu \in \text{supp}(f)\}$ for any $f \in R$.

Let $g = \sum_{i=1}^s a_i \mu_i \in I$, where $\mu_i \in \text{Mon}(S)$ and $a_i \in K \setminus \{0\}$. Call $\mu_i = X_1^{u_{i1}} \cdots X_n^{u_{in}}$. If $\text{Tr}(\text{in}_w(g)) \neq 0$, then there exists $i \in \{1, \dots, s\}$ such that $w(\mu_i) = w(g)$ and $u_{ij} \equiv -1 \pmod{p}$ for all $j = 1, \dots, n$.

Then $\text{Tr}(\text{in}_w(g)) = \sum_{k \in A} \sqrt[p]{a_k} \text{Tr}(\mu_k)$ where $A = \{k \in \{1, \dots, s\} : w(\mu_k) = w(g) \text{ and } u_{kj} \equiv -1 \forall j = 1, \dots, n\}$. By our assumption A is nonempty, indeed $i \in A$. On the other hand, $\text{Tr}(g) = \sum_{k \in B} \sqrt[p]{a_k} \text{Tr}(\mu_k)$ where $B = \{k \in \{1, \dots, s\} : u_{kj} \equiv -1 \forall j = 1, \dots, n\}$. Of course $A \subset B \subset \{1, \dots, s\}$. Furthermore, using (3.1), $\{k \in B : w(\text{Tr}(\mu_k)) \text{ is maximal}\} = \{k \in B : w(\mu_k) \text{ is maximal}\} = A$, so $\text{in}_w(\text{Tr}(g)) = \sum_{k \in A} \sqrt[p]{a_k} \text{Tr}(\mu_k) = \text{Tr}(\text{in}_w(g))$. \square

Corollary 3.10. *Let $f \in S$ be such that there is a monomial order $<$ with $\text{in}_<(f) = X_1^{p-1} \cdots X_n^{p-1}$. Then $f \star \text{Tr}$ is an F -splitting of S , and $\text{in}_<(I) \subset S$ is a squarefree monomial ideal for any compatibly split ideal (with respect to $f \star \text{Tr}$) $I \subset S$.*

Proof. By Proposition 3.8, it is enough to show that $\text{in}_<(I)$ is a compatibly split ideal with respect to $X_1^{p-1} \cdots X_n^{p-1} \star \text{Tr}$. Notice that $\text{in}_<(I)$, as an S -submodule of F_*S , is generated by finitely many monomials, say μ_1, \dots, μ_k ; so to check that $\text{in}_<(I)$ is compatibly split with respect to $X_1^{p-1} \cdots X_n^{p-1} \star \text{Tr}$ it is enough to check that $\text{Tr}(X_1^{p-1} \cdots X_n^{p-1} \mu_i) \in \text{in}_<(I)$ for all $i = 1, \dots, k$. By definition of initial ideal, for any $i = 1, \dots, k$ there are $g_i \in I$ such that $\text{in}_<(g_i) = \mu_i$. Pick a weight vector $w \in (\mathbb{N}_{\geq 0}^n)$ such that $\text{in}_w(g_i) = \text{in}_<(g_i)$ for any $i = 1, \dots, k$ and $\text{in}_w(f) = \text{in}_<(f)$ (so $\text{in}_w(I) = \text{in}_<(I)$). Then either $\text{Tr}(X_1^{p-1} \cdots X_n^{p-1} \mu_i) = 0$ or, using Proposition 3.9,

$$\begin{aligned} \text{Tr}(X_1^{p-1} \cdots X_n^{p-1} \mu_i) &= \text{Tr}(\text{in}_w(f) \text{in}_w(g_i)) \\ &= \text{Tr}(\text{in}_w(f g_i)) \\ &= \text{in}_w(\text{Tr}(f g_i)) \in \text{in}_w(\text{Tr}(f I)) \subseteq \text{in}_w(I) = \text{in}_<(I). \end{aligned}$$

\square

We end this subsection recalling the following useful criterion (see [18, Lemma 1.6]).

Proposition 3.11. *For any $f \in S$ and any ideal $I \subset S$, we have:*

$$(f \star \text{Tr})(I) \subseteq I \iff f \in I^{[p]} : I.$$

3.2. Conclusions. In this subsection we gather the conclusions we can get from the previous subsection. We will not assume anymore that K is a perfect field of positive characteristic. It is useful to recall that, if $\phi : A \rightarrow B$ is a flat homomorphism of Noetherian rings, then for any two ideals $I, J \subset A$ one has (cf. [28, Theorem 7.4]):

$$(I \cap J)B = IB \cap JB, \quad (I : J)B = IB : JB.$$

Theorem 3.12. *Let $S = K[X_1, \dots, X_n]$ be the polynomial ring in n variables over a field K of characteristic $p > 0$. Let $I \subset S$ be an ideal, $<$ a monomial order of S . If $\text{in}_<(I^{[p]} : I)$ contains $X_1^{p-1} \cdots X_n^{p-1}$, then $\text{in}_<(I)$ is a squarefree monomial ideal.*

Proof. Let $f \in I^{[p]} : I$ such that $\text{in}_<(f) = X_1^{p-1} \cdots X_n^{p-1}$. Let K' be the perfect closure of K , $S' = S \otimes_K K'$ and $I' = IS'$. Since the inclusion $S \subset S'$ is flat, then $(I^{[p]} : I)S' = I'^{[p]} : I'$, so $(f \star \text{Tr})(I') \subset I'$ by Proposition 3.11. So, by Corollary 3.10 $\text{in}_<(I')$ is a squarefree monomial ideal. Since the Buchberger algorithm is not affected by field extensions, we conclude that $\text{in}_<(I)$ is a squarefree monomial ideal. \square

The proof of the next result is inspired by the results in [29]. We recall that the m th symbolic power of an ideal I of a Noetherian ring R is the ideal $I^{(m)} := I^m(T^{-1}R) \cap R$ where T is the complement in R of the union of the minimal prime ideals of I . In other words, $r \in I^{(m)}$ if and only if there exists $x \in R$ avoiding all the minimal prime ideals of I such that $rx \in I^m$.

Theorem 3.13. *Let $S = K[X_1, \dots, X_n]$ be the polynomial ring in n variables over a field K (not necessarily of positive characteristic). Let $I \subset S$ be an ideal, $<$ a monomial order of S , and call $h = \max\{\text{ht}(\mathfrak{p}) : \mathfrak{p} \in \text{Min}(I)\}$. If $\text{in}_{<}(I^{(h)})$ contains a squarefree monomial, then $\text{in}_{<}(\sqrt{I})$ is a squarefree monomial ideal.*

Proof. Of course we can assume that I is radical, since $I^{(h)} \subset (\sqrt{I})^{(h)}$.

Let us first assume that K has characteristic $p > 0$. Let $f \in I^{(h)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}^{(h)}$ such that $\text{in}(f)$ is a squarefree monomial. Of course we can assume $\text{in}(f) = X_1 \cdots X_n$, so that $\text{in}(f^{p-1}) = \text{in}(f)^{p-1} = X_1^{p-1} \cdots X_n^{p-1}$. Hence if we show that $f^{p-1} \in I^{[p]} : I$ we are done by Theorem 3.12. Pick $g \in I$. Then $g \in \mathfrak{p}$ for any minimal prime ideal \mathfrak{p} of I . So let us see $g \in \mathfrak{p}S_{\mathfrak{p}}$. Since $S_{\mathfrak{p}}$ is a regular local ring of dimension $\text{ht}(\mathfrak{p}) \leq h$, $\mathfrak{p}S_{\mathfrak{p}}$ is generated by at most h elements. Hence, since $f \in (\mathfrak{p}S_{\mathfrak{p}})^h$, by the pigeonhole principle, then

$$f^{p-1}g \in (\mathfrak{p}S_{\mathfrak{p}})^{[p]}.$$

Hence there exists $a \in S \setminus \mathfrak{p}$ such that $af^{p-1}g \in \mathfrak{p}^{[p]}$. In particular $a^p f^{p-1}g \in \mathfrak{p}^{[p]}$, that is $f^{p-1}g \in \mathfrak{p}^{[p]} : a^p$. So, since S is a regular ring, the Frobenius map $F : S \rightarrow S$ is flat by the theorem of Kunz (cf. [5, Corollary 8.2.8]) $f^{p-1}g \in (\mathfrak{p} : a)^{[p]} = \mathfrak{p}^{[p]}$, and

$$f^{p-1}g \in \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}^{[p]} = \left(\bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p} \right)^{[p]} = I^{[p]}.$$

This concludes the proof if K has positive characteristic.

If K has characteristic 0, let \bar{K} denote the algebraic closure of K , $\bar{S} = \bar{K}[X_1, \dots, X_n]$ and $\bar{I} = \bar{I}\bar{S}$. Since K , having characteristic 0, is perfect, \bar{I} is a radical ideal. Moreover we have an equality of sets $\{\text{ht}(\mathfrak{p}) : \mathfrak{p} \in \text{Min}(\bar{I})\} = \{\text{ht}(\mathfrak{p}) : \mathfrak{p} \in \text{Min}(I)\}$: indeed, given a height c prime ideal $\mathfrak{p} \subset S$, $\mathfrak{p}\bar{S}$ is a (perhaps not prime) ideal of \bar{S} having all the minimal primes of height c , and the prime ideals of $\text{Min}(\bar{I})$ are minimal over some $\mathfrak{p}\bar{S}$ with $\mathfrak{p} \in \text{Min}(I)$. So $h = \max\{\text{ht}(\mathfrak{p}) : \mathfrak{p} \in \text{Min}(\bar{I})\}$. Next, fix $f \in I^{(h)}$ such that $\text{in}(f)$ is a squarefree monomial; so there exists $g \in S \setminus \left(\bigcup_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p} \right)$ such that $fg \in I^h$. Clearly, viewing f and g as polynomials of \bar{S} , we have $fg \in \bar{I}^h$. If g were in some $\mathfrak{p} \in \text{Min}(\bar{I})$, then it would also belong to $\mathfrak{p} \cap S$, which is a minimal prime ideal of I , and we know this is not the case. So $fg \in \bar{I}^h$ and g is not in $\bigcup_{\mathfrak{p} \in \text{Min}(\bar{I})} \mathfrak{p}$, so $f \in \bar{I}^{(h)}$. Therefore, we can assume that K is algebraically closed.

Let $\{f_1, \dots, f_m\}$ be the reduced Gröbner basis of I with respect to $<$. We need to show that $\text{in}(f_i)$ is a squarefree monomial for each $i = 1, \dots, m$.

To this purpose, let us fix $f \in I^{(h)}$ such that $\text{in}(f)$ is a squarefree monomial with coefficient 1 in f ; so there is $g \in S$ such that $fg \in I^h$ and g does not belong to any of the minimal prime ideals of I . We can find a finitely generated \mathbb{Z} -algebra $Z \subset K$ such that the coefficients of f, g , those of all the f_i 's and those of the polynomials of the reduced Gröbner bases of the minimal prime ideals of I are in Z . In particular, if $S_Z = Z[X_1, \dots, X_n]$ and $J_Z = J \cap S_Z$ for any ideal $J \subset S$, we have $I_Z S = I$ and $\mathfrak{p}_Z S = \mathfrak{p}$ for all $\mathfrak{p} \in \text{Min}(I)$.

We also introduce the following notation: for all prime ideals $P \subset Z$, $Q(P)$ denotes the field of fractions of Z/P (we write just Q if P is the zero ideal), $S_{Q(P)} = Q(P)[X_1, \dots, X_n]$ and $J_{Q(P)}$ stands for $J_Z S_{Q(P)}$ for any ideal $J \subset S$. Also, we will write \bar{a} for the image in $S_{Q(P)}$ of an element $a \in S_Z$. Notice that a prime ideal $P \subset Z$ contains at most one prime number $p \in \mathbb{N}$; in this case, $Q(P)$ is a field of characteristic p .

Notice that for any prime number $p > 0$ and for all $P \in \text{Min}(pZ)$ we have that $\{\bar{f}_1, \dots, \bar{f}_m\}$ is a (reduced) Gröbner basis of $I_{Q(P)}$. This is simply because the coefficient of $\text{in}(f_i)$ in $f_i \in S_Z$ is 1

for all $i = 1, \dots, m$, so if the S -polynomials between the f_i 's reduce to zero modulo $\{f_1, \dots, f_m\}$ in S , they reduce to zero modulo $\{\bar{f}_1, \dots, \bar{f}_m\}$ in $S_{Q(P)}$ as well.

Similarly to above, notice that for any prime number $p > 0$ and for all $P \in \text{Min}(pZ)$ we have that $\{\bar{g}_1, \dots, \bar{g}_k\}$ is the reduced Gröbner basis of $\mathfrak{p}_{Q(P)}$ provided that $\{g_1, \dots, g_k\}$ is the reduced Gröbner basis of \mathfrak{p} , for any $\mathfrak{p} \in \text{Min}(I)$. In particular, *for any prime number $p > 0$ and for all $P \in \text{Min}(pZ)$ we have $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p}_{Q(P)})$ and $\bar{g} \notin \mathfrak{p}_{Q(P)}$ for all $\mathfrak{p} \in \text{Min}(I)$.*

We claim that *there exists $N \in \mathbb{N}$ such that, for all prime numbers $p > N$ and $P \in \text{Min}(pZ)$, we have that $I_{Q(P)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}_{Q(P)}$.* The intersection of two polynomial ideals A and B of $S_{Q(P)}$ can be performed by computing a Gröbner basis of $At + B(1-t) \in S_{Q(P)}[t]$, so the claim follows by [29, Lemma 2.5].

We claim that, *for all $\mathfrak{p} \in \text{Min}(I)$, there exists $N_{\mathfrak{p}} \in \mathbb{N}$ such that, for all prime numbers $p > N_{\mathfrak{p}}$ and $P \in \text{Min}(pZ)$, $\mathfrak{p}_{Q(P)}$ is a prime ideal of $S_{Q(P)}$.* To see this, consider the morphism of schemes

$$\phi : X = \text{Spec}(S_Z/\mathfrak{p}_Z) \rightarrow Y = \text{Spec}(Z).$$

Notice that we have that ϕ is of finite type and Y is irreducible. Since $\mathfrak{p}_Q S = \mathfrak{p}_Z S$ is a prime ideal, then the special fibre X_{η} (η is the generic point of Y , namely the zero ideal of Z) is geometrically irreducible and geometrically reduced. Hence by Lemma 37.24.4 of [31, Tag 0574] and Lemma 37.25.5 of [31, Tag 0553] there exists a nonempty open subset $U \subset Y$ such that X_y is geometrically reduced and geometrically irreducible for all $y \in U$. In other words, $\mathfrak{p}_{Q(P)}$ is a geometrically prime ideal of $S_{Q(P)}$ for all prime ideals $P \in U$. We have proved the claim since there exists a nonzero ideal $J \subset Z$ such that $U = \{y \in Y : y \not\supset J\}$, so all but finitely many prime ideals of height 1 in Z belong to U .

Gathering everything, if we pick a prime number $p > \max\{N, N_{\mathfrak{p}} : \mathfrak{p} \in \text{Min}(I)\}$, we proved that any $P \in \text{Min}(pZ)$ is a prime ideal of Z such that:

- $Q(P)$ is a field of characteristic $p > 0$.
- $I_{Q(P)}$ is a radical ideal with $\text{Min}(I_{Q(P)}) = \{\mathfrak{p}_{Q(P)} : \mathfrak{p} \in \text{Min}(I)\}$.
- The maximum height of a minimal prime ideal of $I_{Q(P)}$ is h .
- $\text{in}((I_{Q(P)})^{(h)})$ contains a squarefree monomial.

The above facts, and what previously proved in characteristic $p > 0$, tell us that $\text{in}(I_{Q(P)})$ is a squarefree monomial ideal. That is, $\text{in}(f_i)$ is a squarefree monomial for all $i = 1, \dots, m$, i.e. $\text{in}(I)$ is a squarefree monomial ideal. \square

Example 3.14. *Fix some positive integers $m \leq n$. Let X be an $m \times n$ matrix of indeterminates over a field K of arbitrary characteristic. Let $S = K[X]$ and for $t \leq m$, I_t be the ideal generated by the $t \times t$ minors of the matrix X . It is known that $h := \text{ht}(I_t) = \text{grade}(I_t) = (n-t+1)(m-t+1)$ (see [6, Theorem 2.5]). Following the notation in [6], for indices*

$$1 \leq a_1 < \dots < a_t \leq m \text{ and } 1 \leq b_1 < \dots < b_t \leq n,$$

we set

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] := \det \begin{pmatrix} X_{a_1 b_1} & \cdots & X_{a_1 b_t} \\ \vdots & & \vdots \\ X_{a_t b_1} & \cdots & X_{a_t b_t} \end{pmatrix}.$$

Let

$$f = \left(\prod_{i=t}^{m-1} [1, \dots, i \mid m-i+1, \dots, m][n-i+1, \dots, n \mid 1, \dots, i] \right) \times \prod_{j=1}^{n-m+1} [1, \dots, m \mid j, j+1, \dots, j+m-1].$$

Notice that we have $f \in (\prod_{i=t}^{m-1} I_t^2) I_t^{n-m+1}$; hence $f \in I_t^{(h)}$ [6, Theorem 10.4]. Consider the graded lexicographical ordering $<$ on $K[X]$ with

$$X_{11} > X_{12} > \cdots > X_{1n} > X_{21} > \cdots > X_{m1} > \cdots > X_{mn}.$$

With respect to this ordering, one notices that $\text{in}_{<}(f)$ is squarefree. Hence by Theorem 3.13 we recover the known fact that $\text{in}_{<}(I_t)$ is squarefree.

4. SOME G -DEFORMING F -SINGULARITIES

In this section we will prove that being F -rational or strongly F -injective are G -deforming properties. These facts depend on the fact that these properties deform in the local case by, respectively, [20, Theorem 4.2(h)] and [25, Theorem 4.2(i)]. We show that they also deform in the graded case accordingly with the nonlocal definitions, and to this purpose we proved Theorem 4.3 and Proposition 4.9, that are expected but we could not find in the literature. (We should point out that it would be possible to prove that F -rational or strongly F -injective are G -deforming properties in a more direct way, but we find Theorem 4.3 and Proposition 4.9 interesting by themselves).

A sequence of elements x_1, \dots, x_n in a ring R are called *parameters* if they can be extended to a system of parameters in every local ring $R_{\mathfrak{p}}$ of R where \mathfrak{p} is a prime ideal of R that contains them. An ideal of R is said to be a *parameter ideal* if it can be generated by parameters.

If R has prime characteristic p , the tight closure of an ideal $I \subset R$ is the ideal I^* formed by the elements $r \in R$ such that there exists $c \in R \setminus \bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$ such that $cr^{p^e} \in I^{[p^e]} = (x^{p^e} : x \in I)$ for any positive integer $e \gg 0$. We say that I is tightly closed if $I = I^*$.

Definition 4.1. *A ring of prime characteristic is F -rational if every parameter ideal is tightly closed.*

The following Lemma is well-known. For the convenience of the reader we include a proof.

Lemma 4.2. *Let $f : R \rightarrow S$ be a faithfully flat map. If S is F -rational so is R .*

Proof. Since $R \rightarrow S$ is faithfully flat, then parameters of R go to parameters of S . Let $I \subset R$ be a parameter ideal. Then $(IS)^* = IS$, as S is F -rational. Now $I^*S \subseteq (IS)^* = IS$, hence $I^* = I^*S \cap R \subseteq IS \cap R = I$ (the equalities follow because $R \rightarrow S$ is faithfully flat). So $I^* = I$. \square

Theorem 4.3. *If $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a \mathbb{Z} -graded ring having a unique maximal homogeneous ideal $\mathfrak{m} \subset R$, and (R_0, \mathfrak{m}_0) is a complete local ring, then R is F -rational if and only if $R_{\mathfrak{m}}$ is F -rational.*

Proof. Notice that under the assumptions R is a homomorphic image of a Cohen-Macaulay ring, so the “only if” direction follows from [20, Theorem 4.2(f)]. For the other direction we start by noting that R is a domain since $R_{\mathfrak{m}}$ is a domain.

First let us assume that R_0 is infinite. Then also the multiplicative group $R_0 \setminus \mathfrak{m}_0$ is infinite. Since R is reduced and finitely generated over the excellent local ring R_0 , [32, Theorem 3.5] says that

$$U = \{\mathfrak{p} \in \text{Spec } R : R_{\mathfrak{p}} \text{ is } F\text{-rational}\}$$

is an open subset of $\text{Spec } R$. Let $I \subset R$ be the radical ideal such that $V(I) = \{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supset I\}$ is the complement $\text{Spec } R \setminus U$. Since $\mathfrak{m} \in U$, we are done once we show that I is homogeneous. Consider the action $(R_0 \setminus \mathfrak{m}_0) \times R \rightarrow R$ defined by $\lambda \cdot f = \lambda^d f$ whenever $f \in R_d$, extended by additivity. Because $R_0 \setminus \mathfrak{m}_0$ is infinite, it can be easily checked that I is homogeneous if and only if it is stable under this action. The equivalence in the previous statement can be seen as follows: of course $x \in I$ if and only if R_x is F -rational. Because $\phi_{\lambda} : R \rightarrow R$ sending f to $\lambda \cdot f$ is an automorphism of R for all $\lambda \in R_0 \setminus \mathfrak{m}_0$, R_x is F -rational if and only if $R_{\lambda \cdot x}$ is F -rational

for all $\lambda \in R_0 \setminus \mathfrak{m}_0$. Hence, if $x \in I$, then $\lambda \cdot x \in I$ for any $\lambda \in R_0 \setminus \mathfrak{m}_0$, and this concludes the proof in the case in which $R_0 \setminus \mathfrak{m}_0$ is infinite.

If R_0 is finite, then being a domain, it must be a perfect field. Hence $R_0 \hookrightarrow L$ is a separable extension where L is an algebraic closure of R_0 . Consider $R' = R \otimes_{R_0} L$. Then $R_{\mathfrak{m}} \rightarrow R'_{\mathfrak{n}}$, where \mathfrak{n} is the only maximal homogeneous ideal of R' , is a faithfully flat smooth extension, so by [32, Theorem 3.1], $R'_{\mathfrak{n}}$ is F -rational. Now, since $L = R'_0$ is infinite, by what has been previously said, R' is F -rational, and therefore R is F -rational by Lemma 4.2. \square

Proposition 4.4. *Being F -rational is a G -deforming property.*

Proof. Let R be a \mathbb{N} -graded ring with $R_0 = K$, and suppose that $x \in R$ is a nonzero-divisor on R of degree 1 such that R/xR is F -rational. If $\mathfrak{m} = \bigoplus_{i \geq 1} R_i$, then $R_{\mathfrak{m}}/xR_{\mathfrak{m}}$ is F -rational. So $R_{\mathfrak{m}}$ is F -rational by [20, Theorem 4.2(h)] and hence R is F -rational by Theorem 4.3. By [5, Proposition 10.3.10], R_x is F -rational. This proves condition (1) of the G -deforming definition. Condition (2) of the G -deforming definition follows by Lemma 4.2. \square

Corollary 4.5. *Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K of positive characteristic and $w \in (\mathbb{N}_{>0})^n$. If $I \subset S$ is an ideal such that $S/\text{in}_w(I)$ is F -rational, then S/I is F -rational.*

Proof. This follows from Propositions 2.3 and 4.4 \square

If in the above corollary we replace the word “ F -rational” with “ F -regular” (that is, in all the localizations of R , every ideal is tightly closed), the statement is false.

Example 4.6. *Let $S = K[X_1, \dots, X_5]$ where K has characteristic $p > 2$, and I the ideal generated by the 2-minors of the matrix:*

$$\begin{pmatrix} X_4^2 + X_5^3 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

Note that, if $\deg(X_4) = 3$, $\deg(X_1) = \deg(X_3) = 6$, $\deg(X_2) = 24$ and $\deg(X_5) = 2$, the ideal I is homogeneous. By [30, Proposition 4.5] S/I is not F -regular. However, considering the weight vector $w = (6, 24, 6, 3, 1)$ of $(X_1, X_2, X_3, X_4, X_5)$, one has that $\text{in}_w(I)$ is the ideal of 2-minors of the matrix:

$$\begin{pmatrix} X_4^2 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

By [30, Proposition 4.3] $S/\text{in}_w(I)$ is F -regular, so “ F -regularity” is not a G -deforming property.

Next we want to prove that being “ F -full” or “strongly F -injective” are G -deforming properties.

Lemma 4.7. *Let $f : R \rightarrow S$ be a faithfully flat map between homomorphic images of regular rings of prime characteristic. If S is F -full, so is R .*

Proof. First note that since f is faithfully flat the natural map $\text{Spec } S \rightarrow \text{Spec } R$ induced by f is surjective. We will show $R_{\mathfrak{m}}$ is F -full for every maximal ideal \mathfrak{m} of R , that is equivalent to say that R is F -full since $H_{\mathfrak{m}R_{\mathfrak{m}}}^i(R_{\mathfrak{m}}) \cong H_{\mathfrak{m}}^i(R)$.

Let \mathfrak{m} be a maximal ideal of R . Let \mathfrak{n} be a maximal ideal in S containing $\mathfrak{m}S$. Then $R_{\mathfrak{m}} \rightarrow S_{\mathfrak{n}}$ is a flat local map. By hypothesis $S_{\mathfrak{n}}$ is F -full. By [12, Proposition 3.9, Corollary 2.2], $R_{\mathfrak{m}}$ is F -full. \square

Proposition 4.8. *Let $R = S/I$ with S is an n -dimensional regular ring of prime characteristic. Then R is F -full iff the natural map $\text{Ext}_S^i(R, S) \rightarrow H_1^i(S)$ is injective for every $i = 0, \dots, n$.*

In particular, if R is a homomorphic image of a regular ring:

- the F -full locus $\{\mathfrak{p} \in \text{Spec } R : R_{\mathfrak{p}} \text{ is } F\text{-full}\}$ is a Zariski open subset of $\text{Spec } R$.

- If $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is a \mathbb{Z} -graded ring having a unique maximal homogeneous ideal $\mathfrak{m} \subset R$, then R is F -full if and only if $R_{\mathfrak{m}}$ is F -full.

Proof. By definition R is F -full if and only if $R_{\mathfrak{m}}$ is F -full for all maximal ideals $\mathfrak{m} \subset R$, or equivalently, if and only if $R_{\mathfrak{m}}$ is F -full for all maximal ideals $M \subset S$ containing I and $\mathfrak{m} = M/I$. On the other hand, by [12, Proposition 2.1, Corollary 2.2], $R_{\mathfrak{m}}$ is F -full if and only if the natural map

$$\mathrm{Ext}_S^i(R, S)_M \cong \mathrm{Ext}_{S_M}^i(R_{\mathfrak{m}}, S_M) \rightarrow H_{IS_M}^i(S_M) \cong H_I^i(S)_M$$

is injective for every $i = 0, \dots, n$. Clearly, the above maps are injective for all maximal ideals $M \subset S$ containing I and for all $i = 0, \dots, n$ if and only if the maps $\mathrm{Ext}_S^i(R, S) \rightarrow H_I^i(S)$ are injective for all $i = 0, \dots, n$.

For the last part, just call N_i the kernel of $\mathrm{Ext}_S^i(R, S) \rightarrow H_I^i(S)$. Then, for a prime ideal $\mathfrak{p} \in \mathrm{Spec} R$, $R_{\mathfrak{p}}$ is F -full if and only if $(N_i)_{\mathfrak{p}} = 0$ for all $i = 0, \dots, n$. Therefore the F -full locus of R is $\mathrm{Spec} R \setminus \bigcup_{i=0}^n \mathrm{Supp} N_i$, that is open. Finally, in the graded case, N_i is a graded R -module, so $(N_i)_{\mathfrak{m}} = 0$ implies $N_i = 0$. \square

Proposition 4.9. *Being F -full or strongly F -injective are G -deforming properties.*

Proof. Let (R, \mathfrak{m}) be an \mathbb{N} -graded ring with $R_0 = K$. Suppose that R/xR is F -full (strongly F -injective) for some homogeneous element x of degree 1. Then $R_{\mathfrak{m}}/xR_{\mathfrak{m}}$ is F -full (strongly F -injective) by definition. Hence $R_{\mathfrak{m}}$ is F -full (strongly F -injective) by [25, Theorem 4.2, Corollary 5.16], thus R is F -full (strongly F -injective) by Proposition 4.8 and [13, Theorem 5.12]. By [12, Lemma 3.4] and [13, Theorem 3.3], R_x is F -full (strongly F -injective), so condition (1) of G -deforming property definition is satisfied. Now condition (2) follows from Lemma 4.7 and [13, Theorem 3.9]. \square

Thus similar to Corollary 4.5, we have

Corollary 4.10. *Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K of positive characteristic and $w \in (\mathbb{N}_{>0})^n$. If $I \subset S$ is an ideal such that $S/\mathrm{in}_w(I)$ is strongly F -injective, then S/I is strongly F -injective.*

Corollary 4.11. *Let K be a field of characteristic $p > 0$ and $<$ a monomial order on $S = K[X_1, \dots, X_n]$, $I \subset S$ an ideal of S and $A \subset S$ a K -subalgebra.*

- (1) *If $\mathrm{in}_{<}(A)$ is Noetherian and normal, then A is F -rational.*
- (2) *If $\mathrm{in}_{<}(I)$ is radical, then S/I is strongly F -injective, and so F -injective.*

Proof. (1). If $f_1, \dots, f_m \in A$ are a Sagbi basis of A , that is $\mathrm{in}_{<}(A) = K[\mathrm{in}_{<}(f_1), \dots, \mathrm{in}_{<}(f_m)]$, it is easy to see that $A = K[f_1, \dots, f_m]$. It turns out that, if $J \subset P = K[Y_1, \dots, Y_m]$ is the kernel of the K -algebra homomorphism sending Y_i to f_i (so that $P/J \cong A$), there exists $u \in (\mathbb{N}_{>0})^m$ such that $\mathrm{in}_u(J)$ is the kernel of the K -algebra homomorphism sending Y_i to $\mathrm{in}_{<}(f_i)$ (hence $\mathrm{in}_u(J)$ is a binomial ideal and $P/\mathrm{in}_u(J) \cong \mathrm{in}_{<}(A)$). Since $P/\mathrm{in}_u(J) \cong \mathrm{in}_{<}(A)$ is a normal toric ring, it is F -regular, being a direct summand of a polynomial ring ([20, Proposition 4.12]). In particular, $P/\mathrm{in}_u(J)$ is F -rational. So, by Proposition 4.5, $A \cong P/J$ is F -rational.

(2). Since $\mathrm{in}_{<}(I)$ is radical, it is generated by square free monomials, hence $S/\mathrm{in}_{<}(I)$ is F -split, in particular strongly F -injective, hence by Corollary 4.10, S/I is strongly F -injective. \square

Remark 4.12. *The conclusion that A is F -rational if $\mathrm{in}_{<}(A)$ is normal was already proved in [9, Corollary 2.3]. In the proof is used that a \mathbb{N} -graded ring R (with R_0 a field of positive characteristic) is F -rational whenever R/xR is so for some non-zero divisor $x \in R_1$. It is not clear to us how to show this fact (certainly known in the local case) without using Theorem 4.3.*

The conclusion that S/I is F -injective provided $\mathrm{in}_{<}(I)$ is a squarefree monomial ideal has been proved independently in [19, Theorem 4.3], where this result has been crucial to prove that the only Gorenstein binomial edge ideals are complete intersections.

Example 4.13. Let $S = K[X_1, \dots, X_5]$ where K has characteristic $p > 3$, and I the ideal generated by the 2-minors of the matrix of Example 4.6, namely:

$$\begin{pmatrix} X_4^2 + X_5^3 & X_3 & X_2 \\ X_1 & X_4^2 & X_3^4 - X_2 \end{pmatrix}.$$

If $<$ is the lexicographic monomial order with $X_1 > X_2 > X_3 > X_4 > X_5$, then $\text{in}_<(I) = (X_1X_3, X_1X_2, X_2X_3)$, so S/I is strongly F -injective by Corollary 4.11. However, S/I is not F -split by [30, Proposition 4.5].

The following corollary can help in recognising certain classes of projective varieties whose defining ideal, in any embedding, cannot admit a squarefree initial ideal.

Corollary 4.14. Let X be a projective scheme over a field K of characteristic 0 such that, for some embedding of X in \mathbb{P}^n and monomial order $<$ on $K[x_0, \dots, x_n]$, we have that $\text{in}_<(I)$ is squarefree (where I is the defining ideal of the embedding). Then the Frobenius action on $H^i(X_p, \mathcal{O}_{X_p})$ must be injective for all $i > 0$ and prime number $p \gg 0$ (X_p denotes a reduction mod p of X).

Proof. Let $X \simeq \text{Proj } S/I$, where $S = K[x_0, \dots, x_n]$ with respect to some embedding of X in \mathbb{P}^n with defining ideal $I = (f_1, \dots, f_t)$. We can, and will, choose f_1, \dots, f_t forming a Gröbner basis. Choose a finitely generated \mathbb{Z} -algebra A in such a way that, taking $S_A = A[x_0, \dots, x_n]$ and defining $I_A = (f_1, \dots, f_t)S_A$, we have that S_A/I_A is free over A and $S_A/I_A \otimes_A K = S/I$. Let $X_A = \text{Proj } S_A/I_A$. Then a reduction modulo a prime number p of X has the form $X_p = \text{Proj } S_A/I_A \otimes L$ where $L = A/P$ for a maximal ideal $P \subset A$ containing p . In particular $S_A/I_A \otimes L = S_p/I_p$, where $S_p = L[x_1, \dots, x_n]$ where L is a field of characteristic $p > 0$ and $I_p = (\bar{f}_1, \dots, \bar{f}_t)$. Furthermore, if p is big enough, we can assume $\{\bar{f}_1, \dots, \bar{f}_t\}$ remains a Gröbner basis of I_p and $\text{in}(\bar{f}_i) = \text{in}(f_i)$ for all i . Hence $\text{in}(I_p)$ is also square free. Thus by Corollary 4.11, S_p/I_p is F -injective for all $p \gg 0$. Since for all $i > 0$, $H^i(X_p, \mathcal{O}_{X_p}) = [H_{\mathfrak{m}_p}^{i+1}(S_p/I_p)]_0$, where \mathfrak{m}_p denotes the homogeneous maximal ideal of S_p/I_p , the Frobenius action on $H^i(X_p, \mathcal{O}_{X_p})$ is injective. \square

5. EXAMPLES

For the convenience of the reader we recall briefly the definitions of *Algebra with Straightening Law (ASL)* and *Cartwright-Sturmfels ideals*. For more details see, respectively, [6] and [7].

ASL. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a \mathbb{N} -graded algebra and let $(H, <)$ be a finite poset. Let $H \rightarrow \bigcup_{i > 0} A_i$ be an injective function. The elements of H will be identified with their images. Given a chain $h_1 \preceq h_2 \preceq \dots \preceq h_s$ of elements of H the corresponding product $h_1 \cdots h_s \in A$ is called standard monomial. One says that A is an ASL on H (with respect to the given embedding H into $\bigcup_{i > 0} A_i$) if three conditions are satisfied:

- The elements of H generate A as a A_0 -algebra.
- The standard monomials are A_0 -linearly independent.
- For every pair h_1, h_2 of incomparable elements of H there is a relation

$$h_1 h_2 = \sum_{j=1}^u \lambda_j h_{j1} \cdots h_{jv_j}$$

where $\lambda_j \in A_0 \setminus \{0\}$, the $h_{j1} \cdots h_{jv_j}$ are distinct standard monomials and, assuming that $h_{j1} \preceq \dots \preceq h_{jv_j}$, one has $h_{j1} < h_1$ and $h_{j1} < h_2$ for all j .

Cartwright-Sturmfels ideals. Given positive integers d_1, \dots, d_m one considers the polynomial ring $S = K[X_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq d_i]$ with \mathbb{Z}^m -graded structure induced by assignment

$\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. The group $G = \mathrm{GL}_{d_1}(K) \times \cdots \times \mathrm{GL}_{d_m}(K)$ acts on S as the group of multigraded K -algebra automorphisms. The Borel subgroup $B = B_{d_1}(K) \times \cdots \times B_{d_m}(K)$ of the upper triangular invertible matrices acts on S by restriction. An ideal J is Borel-fixed if $g(J) = J$ for all $g \in B$. A multigraded ideal $I \subset S$ is Cartwright-Sturmfels if its multigraded Hilbert function coincides with that of a Borel-fixed radical ideal.

Corollary 5.1. *Let K be a field of characteristic $p > 0$. Then the following K -algebras are strongly F -injective, and so F -injective:*

- (1) *Algebras with straightening law.*
- (2) *Quotients of the form S/I where S is a polynomial ring over K and I is a Cartwright-Sturmfels ideal (e.g. I is a binomial edge ideal).*

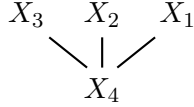
Proof. (1) Writing $A = S/I$ where S is a polynomial ring in variables indexed by the poset H over K , and I is the ideal generated by the straightening relations, choosing a degree (according to the grading of A) reverse lexicographic order $<$ extending the partial order on H , it easily follows from the definition that $\mathrm{in}_{<}(I)$ is a quadratic squarefree monomial ideal, hence A is strongly F -injective by Corollary 4.11.

(2) In [7, Proposition 1.6] it has been shown that, in this case, $\mathrm{in}_{<}(I)$ is a squarefree monomial ideal for any monomial order, so the thesis follows once again by Corollary 4.11. \square

Remark 5.2. *Let $S = K[X_1, X_2, X_3, X_4]$, where K is algebraically closed field of characteristic $p > 0$, and I the ideal generated by the 2-minors of the matrix:*

$$\begin{pmatrix} X_4^4 & X_1 & X_3 \\ X_2 & X_4^4 & X_2 - X_3 \end{pmatrix}.$$

One notes that $I = (X_1X_2 - X_4^8, X_2X_3 - X_4^4(X_2 - X_3), X_1X_3 - X_4^8 + X_4^4X_3)$. It is easy to check that the ring S/I is an ASL on the poset H below:



that is, in the poset H we have $X_4 < X_3, X_2, X_1$ (X_1, X_2 and X_3 are incomparable). By [21, Example 7.15] S/I is F -rational but not F -split. In particular, there are algebras with straightening law that are not F -split. Notice that the poset H is “wonderful” in the terminology of [15], and for an ASL A with A_0 a complete local ring being F -split and F -pure are equivalent conditions, so this is a counterexample to a conjecture stated at page 245 of [15].

Similarly, there are Cartwright-Sturmfels ideals which are not F -pure. For example consider the binomial edge ideal of a pentagon, namely

$$I = (X_iY_{i+1} - X_{i+1}Y_i, X_5Y_1 - X_1Y_5 \quad : \quad i = 1, 2, 3, 4) \subset S = K[X_1, \dots, X_5, Y_1, \dots, Y_5].$$

We have that I is a Cartwright-Sturmfels ideal by [8, Theorem 2.1], however, if K has characteristic 2, S/I is not F -split by [27, Example 2.7].

Corollary 5.3. *Let K be a field of characteristic $p > 0$. Then the following K -algebras are F -split:*

- *Gorenstein ASL.*
- *Gorenstein quotients of the form S/I where S is a polynomial ring over K and I is a Cartwright-Sturmfels ideal*

Proof. For a Gorenstein ring being F -split is equivalent to being F -injective by [18, Lemma 3.3]. So the result follows from Corollary 5.1. \square

The following argument has been suggested by Winfried Bruns.

Corollary 5.4. *Let $M_t(X)$ be the set of t -minors of a $m \times n$ generic matrix X , and K a field of characteristic $p > \min\{t, m - t, n - t\}$. The algebra of minors $K[M_t(X)]$ is F -regular.*

Proof. First of all, by [2, Theorem 3.11] there exists a monomial order such that $\text{in}(K[M_t(X)])$ is a normal semigroup ring, so $K[M_t(X)]$ is F -rational by Corollary 4.11.

In order to see that $K[M_t(X)]$ is F -regular, we assume that $m \leq n$. So, let us add $n - m$ rows to X in order to form the generic $n \times n$ -matrix X' . By [4, Proposition 1.4] and [20, Proposition 4.12], if $K[M_t(X')]$ is F -regular, then $K[M_t(X)]$ is F -regular as well. Furthermore, by [4, Proposition 1.3], $K[M_t(X')]$ is F -regular if and only if $K[M_{n-t}(X')]$ is F -regular.

So we can assume that X is a generic $n \times n$ -matrix and $t \geq n/2$. Now we can add $2t - n$ rows and $2t - n$ columns to X and get a generic $2t \times 2t$ -matrix X' . Again using [4, Proposition 1.4] and [20, Proposition 4.12], if $K[M_t(X')]$ is F -regular, then $K[M_t(X)]$ is F -regular as well.

So, we can eventually assume that X is a generic $2t \times 2t$ -matrix. In this case, $K[M_t(X)]$ is Gorenstein by [3, Theorem 5.5]. Since a Gorenstein ring is F -regular if and only if it is F -rational by [20, Corollary 4.7(a)], we are done. \square

Remark 5.5. *When $t = \min\{m, n\}$, the K -algebra $K[M_t(X)]$ is the coordinate ring of a Grassmannian in its Plücker embedding, and in this case the F -regularity in positive characteristic had already been proved in [21, Theorem 7.14].*

In general, that the K -algebra $K[M_t(X)]$ is F -rational whenever K a field of characteristic $p > \min\{t, m - t, n - t\}$ was already known and proved in [1]. Analogously, in [1] it has been proved that also the Rees algebra of the ideal of the t -minors of X is F -rational whenever K a field of characteristic $p > \min\{t, m - t, n - t\}$. The F -split and F -regularity properties for these and other blowup algebras of determinantal objects are studied in [14].

We conclude with the following corollary, recently proved in [11, Theorem 4.3].

Corollary 5.6. *If X is a smooth projective curve of genus 1 over the rationals, then $\text{in}_<(I)$ is never squarefree, where I is the homogeneous ideal defining $X \subset \mathbb{P}^n$ (independently on the embedding).*

Proof. Since X is a smooth curve of genus 1, then X is isomorphic to an elliptic curve. Then for infinitely many primes p , the reduction mod p considered in [17], X_p of X is supersingular [17, Theorem 1]. That is the Frobenius morphism on $H^1(X_p, \mathcal{O}_{X_p})$ is zero for infinitely many primes p . Hence the corollary follows from Corollary 4.14. \square

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