Singularities of ideals admitting a squarefree Gröbner degeneration

Colloquium Lecture at SLMath

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People involved (so far) in this project

Aldo Conca, Alexandru Constantinescu, Emanuela De Negri, Amy Huang, Mitra Koley, Jonah Tarasova, Emily Witt.

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2. $I \subset S$ homogeneous ideal.
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Goal: understand the singularities of S/I.

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Determinantal (generic, symmetric, skew-symmetrc, Hankel) varieties, Grassmannians, matrix Schubert varieties, Veronese embeddings, rational normal scrolls, ASL...

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A general fact:

Koley-_, González-Martinez

If $\mathrm{char}(K) = p > 0$, then S/I is F-injective.

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In general, it is still unknown whether F-injectivity deforms.

If $\mathrm{char}(K) = p > 0$, it may happen that S/I is not F-pure:

Examples

Ideals $J \subset S$ such that $\mathrm{in}_<(J)$ is squarefree but S/J is not F-pure:

- Fedder (1983): quasi-homogeneous in small characteristic.
- Hochster-Huneke (1994), Singh (1999): quasi-homogeneous in all (high enough) characteristics.
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Also, there are homogeneous ideals $J \subset S$ such that S/J is F-pure but $\mathrm{in}(J)$ is not squarefree for any monomial order on S:

Example

If $J = (X^3 + Y^3 + Z^3) \subset S = K[X, Y, Z]$, S/I is F-pure whenever $\mathrm{char}(K) = p \equiv 1 \pmod 3$, but $\mathrm{in}(J) \in \{(X^3), (Y^3), (Z^3)\}$, so it is not squarefree for any monomial order.

However, Fedder's criterion tells us that, if $J \subset S$ is homogeneous and $\mathrm{char}(K) = p > 0$, S/J is F-pure if and only if there exists $f \in J^{[p]} : J$ such that $X_1^{p-1} \cdots X_n^{p-1} \in \mathrm{supp}(f)$.

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Koley-_

If $X_1^{p-1} \cdots X_n^{p-1} = \mathrm{in}_<(f)$, then $\mathrm{in}_<(J)$ is squarefree.

We also proved a characteristic-free version of the previous result:

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Over any field K, if $J \subset S$ is a radical unmixed ideal of height h, then $\mathrm{in}_<(J)$ is squarefree as soon as there exists $g \in J^{(h)}$ such that $\mathrm{in}_<(g)$ is a squarefree monomial.

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In particular, if there is a regular sequence $f_1, \ldots, f_h \subset J$ such that $\mathrm{in}_<(f_1), \ldots, \mathrm{in}_<(f_h)$ is a squarefree regular sequence, $\mathrm{in}_<(J)$ is squarefree (since $f = f_1 \cdots f_h \in J^h \subset J^{(h)}$ has a sqf initial term).

In our setting, $\mathrm{in}_<(I)$ is squarefree, so it is the Stanley-Reisner ideal of some simplicial complex Δ on n vertices:

$$\mathrm{in}_<(I) = I_\Delta = (X_{i_1} \cdots X_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta) \subset S.$$

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In the same paper, we raised a question, later formulated as conjecture with Constantinescu and De Negri:

Conjecture (Constantinescu-De Negri-_)

If X is smooth and irreducible, then S/I is Cohen-Macaulay with negative a-invariant.

Comment

When X is smooth, exploiting the previous result with Conca, one can show that the following conditions are equivalent:

1. S/I is Cohen-Macaulay with negative a-invariant.
2. If $\operatorname{char}(K) = 0$, S/I is a rational singularity.
3. If $\operatorname{char}(K) = p > 0$, S/I is F-rational.
4. S/I_Δ is Cohen-Macaulay with negative a-invariant.
5. Δ is acyclic.

Let $n \geq 2$, $f \in S = K[X_1, \ldots, X_n]$ a homogeneous polynomial of degree n. If $\mathrm{in}(f)$ is squarefree, it is easy to see that f cannot define an irreducible smooth hypersurface:

Example

Let $f = X_1^3 + X_2^3 + X_3^3 + X_1X_2X_3 \in K[X_1, X_2, X_3]$. Then

$$\mathrm{in}(f) \in \{X_1^3, X_2^3, X_3^3\}.$$

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The conjecture is open even for curves, in which case it can be rephrased as:

Conjecture

If X is an irreducible smooth curve, then X has genus 0.

Comment

There exist irreducible smooth projective curves of genus > 0 with F-injective coordinate ring.

Constantinescu-De Negri-_

If $K = \mathbb{Q}$ and $J \subset S$ is a homogeneous ideal such that $C = \mathrm{Proj}(S/J)$ is a smooth curve of genus 1, then $\mathrm{in}(J)$ cannot be squarefree.

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This kind of proofs cannot work in positive characteristic, and not even if $K = \mathbb{C}$.

The conjecture is known also if $<$ is a **degrevlex** monomial order. In this case it is also true the following stronger version:

Constantinescu-De Negri-_

If $<$ is a **degrevlex** monomial order and X is an irreducible variety with rational (F-rational) singularities then S/I is a rational (F-rational) singularity.

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Corollary

If $\operatorname{char}(K) = p > 0$ and $f \in S$ is a polynomial with squarefree initial monomial w.r.t. degrevlex, for every compatibly split homogeneous ideal J w.r.t. the Frobenius splitting given by f in $\mathrm{Hom}_S(F^*S, S)$ we have: $\mathrm{Proj}(S/J)$ smooth $\Rightarrow S/J$ is CM.

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If $X = \mathrm{Proj}(S/I)$ is an irreducible smooth curve, then Δ is a tree.

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We will approach this conjecture by starting with a graph Δ which is not a tree and finding obstructions to a **Gröbner smoothing** of it:

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We will approach this conjecture by starting with a graph Δ which is not a tree and finding obstructions to a **Gröbner smoothing** of it: i.e. to the existence of a homogeneous ideal $J \subset S$ such that $\mathrm{Proj}(S/J)$ is smooth and $\mathrm{in}_<(J) = I_\Delta$.

Example

If J is the ideal of 2-minors of the $2 \times (n-1)$-matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_{n-1} \\ X_2 & X_3 & \dots & X_n \end{pmatrix},$$

$\mathrm{Proj}(S/J)$ is a rational normal curve (hence smooth) and choosing **lex** with $X_1 > X_2 > \ldots > X_n$ we have:

$$\mathrm{in}_<(J) = (X_i X_j : i = 1, \ldots, n-2, i+2 \le j \le n).$$

Hence $\mathrm{in}_<(J)$ is squarefree.

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Hence $\mathrm{in}_<(J)$ is squarefree. In this case, $\mathrm{in}_<(J) = I_\Delta$ where Δ is:

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The idea is, assuming WLOG that X_1 is the biggest variable, to show that, whenever $J \subset S$ is a homogeneous ideal such that $\mathrm{in}_<(J) = I_\Delta$, the corresponding point $P_1 = [1 : 0 : \ldots : 0] \in \mathbb{P}^{n-1}$

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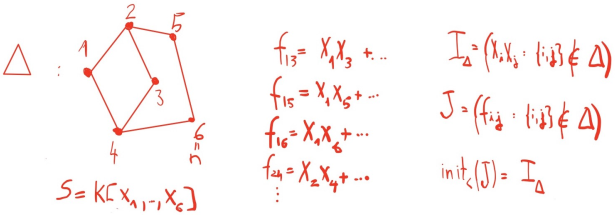

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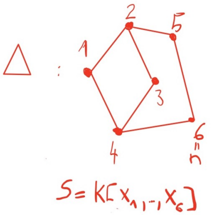

$f_{13} = X_1X_3 + \dots$

$f_{15} = X_1X_5 + \dots$

$f_{16} = X_1X_6 + \dots$

$f_{24} = X_2X_4 + \dots$

$\vdots$

$I_\Delta = (X_iX_j : \{i,j\} \notin \Delta)$

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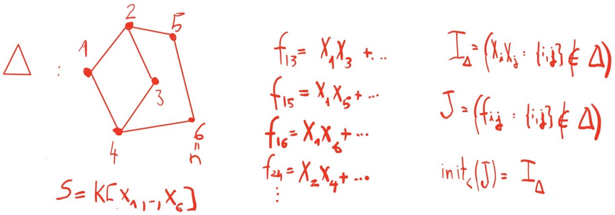

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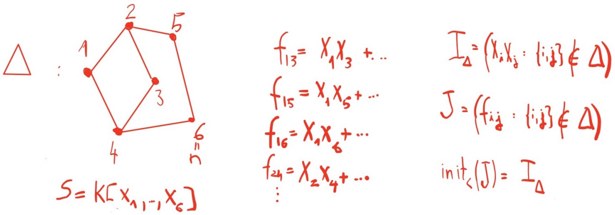

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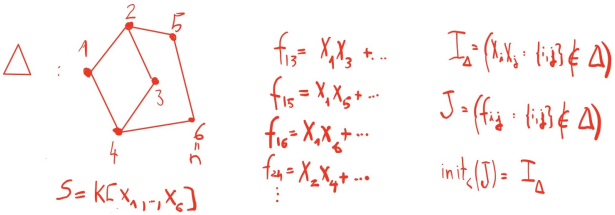

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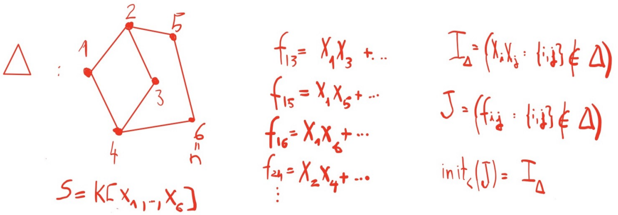

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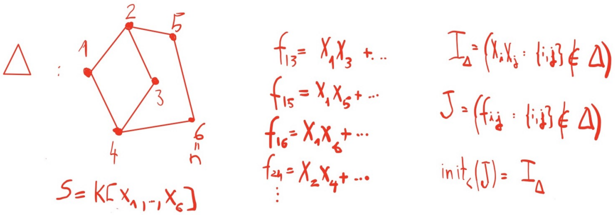

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Without changing variables P_i may not be a point of $\mathrm{Proj}(S/J)$. However most change of variables are not allowed, in the sense that the initial ideal will change.

If Δ has leaves the situation is more complicated: indeed if the biggest variable X_1 is a leaf, the point P_1 will be a smooth point of $\mathrm{Proj}(S/J)$.

With Huang, Tarasova and Witt we are trying to show that the point corresponding to the biggest variable X_i in a cycle of Δ, $P_i \in \mathbb{P}^{n-1}$, will be a singular point of $\mathrm{Proj}(S/J)$ after a suitable change of variables.

Comment

Without changing variables P_i may not be a point of $\mathrm{Proj}(S/J)$. However most change of variables are not allowed, in the sense that the initial ideal will change. This is one reason why standard deformation techniques do not apply for this conjecture, since any change of variables is allowed in that framework.

So far we have:

Huang-Tarasova-Witt-_

If Δ has at most one leaf, then it does not admit a Gröbner smoothing.

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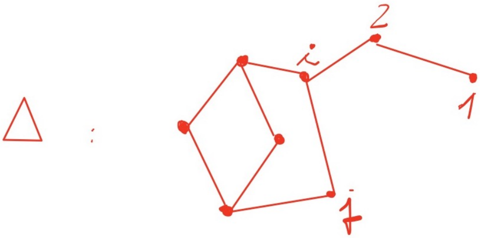
The idea is to use projections and control the singularities: such projections are from a point of the curve and are not generic, so the partial elimination ideals of Green do not help in this framework.



THANKS FOR YOUR ATTENTION!